MTH 252Z Notes

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Contents

1 Prerequisite Review **4**

1.1 Algebra 4

1.2 Derivatives 7

1.2.1 Derivative Formulas 7

2 Antiderivatives **9**

2.1 Antiderivatives 9

2.2 Indefinite Integrals 11

3 Sums & Area **15**

3.1 Sigma Notation 15

3.2 Areas 18

4 Definite Integrals & Average Values **21**

4.1 The Definite Integral 21

4.2 Average Value 24

5 The Fundamental Theorem of Calculus **25**

5.1 The Accumulation Function 25

5.2 The Fundamental Theorem of Calculus 25

5.3 FTC 2 27

5.4 Net Change Theorem 28

5.5 Proof of FTC 30

6 Substitution **31**

6.1 Differentials . . . 31
6.2 Substitution . . . 32
6.3 Substitution with Definite Integrals . . . 34
6.4 Even and Odd Functions . . . 35

7 Integration By Parts **37**

8 Trigonometric Integrals **40**

9 Partial Fraction Decomposition **42**

10 Numerical Integration **44**

10.1 Summary of Formulas for this Section . . . 44
10.2 Rectangle and Trapezoid Approximations . . . 44
10.2.1 Geometric Justifications of the Midpoint and Trapezoid Rules . . . 45
10.3 Error . . . 47
10.4 Simpson's Rule . . . 48
10.5 Establishing Simpson's Rule . . . 49

11 Improper Integrals **50**

11.1 Type 1 Improper Integrals . . . 51
11.2 Type 2 Improper Integrals . . . 52
11.3 The Comparison Theorem . . . 55

12 Area Between Curves **56**

12.1 Building the Definite Integral . . . 56
12.2 Area Between Curves . . . 56
12.3 Horizontally Simple Regions . . . 58

13 Volumes **60**

13.1 Volume by Disk . . . 60
13.2 Volume by Washer . . . 63
13.3 Summary of Formulas for this Section . . . 64

13.4 Volume by Cylindrical Shells 65

14 Arc Length **67**

15 Work **70**

15.1 Introduction to Work 70

15.2 Building a Definite Integral for Work 70

15.3 Springs 71

16 Hydrostatic Force **72**

1 Prerequisite Review

1.1 Algebra

In order to succeed in calculus, we must have a good understanding of algebra. This includes but is not limited to being practiced at

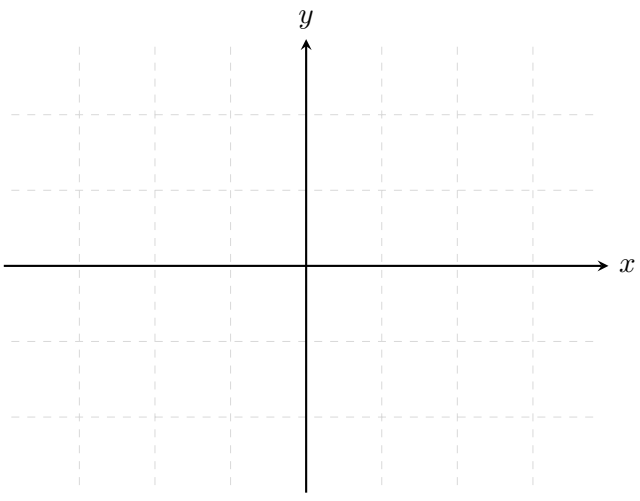
- Factoring
- Expanding
- Graphing
- Solving Equations
- Laws of Exponents
- Laws of Logarithms
- Translating from Radicals to and from Powers

For practice, we will try these.

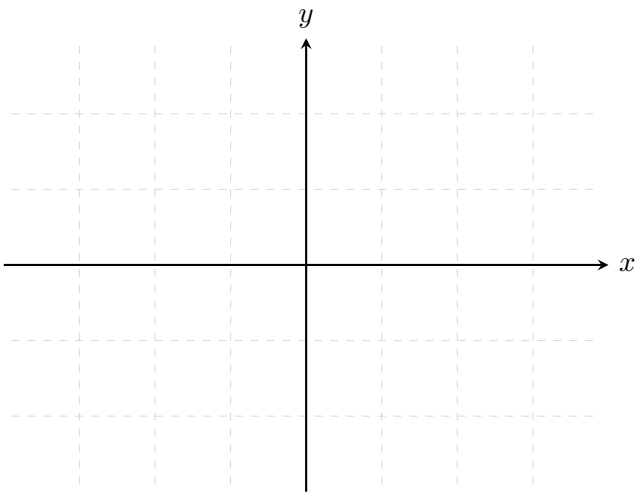
Example 1. Factor $32x^4 - 8x^3 - 60x^2$.

Example 2. Sketch the graph of each function. Be sure to provide a scale on each set of axes.

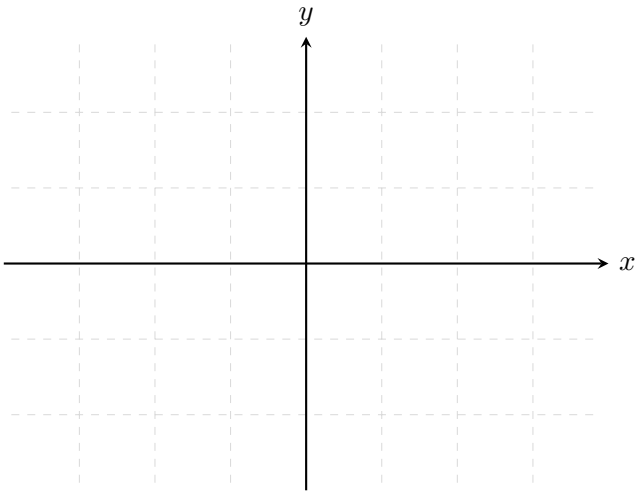
a. $\sin x$



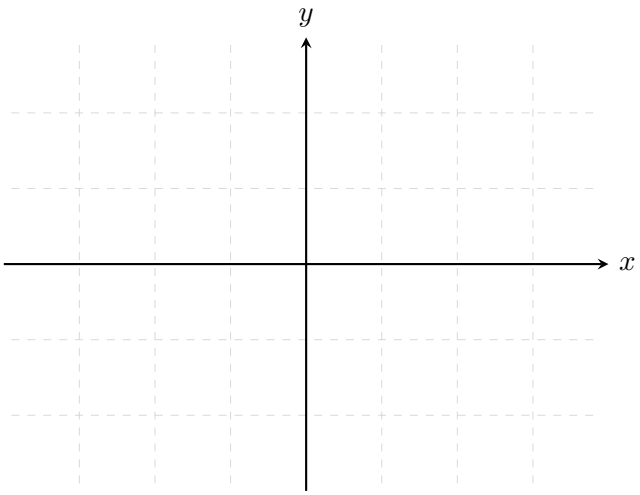
b. $\tan x$



c. $\ln x$



d. e^x



Example 3. Solve the equation $8x^2 - 14x + 12 = 0$.

Example 4. Solve the equation $(e^x - 2)(\ln x + 2) = 0$.

1.2 Derivatives

If $y = f(x)$, then $y' = \frac{dy}{dx} = f'(x)$ all represent the **derivative of f (or y) with respect to x**. The notation $\frac{dy}{dx}$ is in **Leibniz notation**, and $f'(x)$ is in **prime notation**. Generally, the following are the same:

- The derivative of f at a.
- The instantaneous rate of change of f at a.
- The slope of the line tangent to the curve $y = f(x)$ at $(a, f(a))$.

We know that if $y = f(x)$, then $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$. Typically, we do not use this to compute the derivative. Instead, we have a number of derivative rules and derivative formulas. We will need these in order to progress through this course.

If f and g are differentiable (and in the case of the chain rule, if f is differentiable at $g(x)$), then

Name	Rule
Sum Rule	$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$
Difference Rule	$\frac{d}{dx}\left(f(x) - g(x)\right) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$
Product Rule	$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{d}{dx}(f(x))g(x) + f(x)\frac{d}{dx}(g(x))$
Quotient Rule	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$
Constant Multiple Rule	$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$
Chain Rule	$\frac{d}{dx}(f(g(x))) = f'(g(x))\frac{d}{dx}(g(x))$
General Power Rule	$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

1.2.1 Derivative Formulas

The derivative rules are great, but in order to use them, we need a catalog of derivatives of specific functions. That's where the following list comes in.

Function	Derivative
c	0
e^x	e^x
a^x	$\ln(a)a^x$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{\ln(a)x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$

Ex: Find an equation for the line tangent to $y = \cos(2x) - \arctan(x)$ at $\left(\frac{\pi}{4}, -1\right)$. Round your slope to the nearest tenth.

2 Antiderivatives

2.1 Antiderivatives

Definition

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Suppose $f(x) = x^2$. We want to find a function whose derivative is x^2. Let's explore using Guess-and-Check.

Guess #1:

Check:

Guess #2:

Check:

Guess #3:

Check:

Guess #4:

Check:

Corollary

If $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$, where C is constant.

This corollary (to the Mean Value Theorem) states that once you find an antiderivative, you do not need to find any others.

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.

Returning to $f(x) = x^2$, $F(x) = \frac{1}{3}x^3 + C$ represents a **family** of functions whose graphs are vertical translations of each other. This makes sense, for each curve has the same slope at any given x.

Example 1. Find the most general antiderivative of each function below.

$$f(x) = \sin x \qquad g(x) = \frac{1}{x} \qquad p(x) = x^n, n \neq 1$$

2.2 Indefinite Integrals

Definition

The operation that takes a function f and outputs the family of antiderivatives, F, is called an **indefinite integral** of f. We write

$$\int f(x)\ dx = F(x)$$

Thus, $\int f(x)\ dx = F(x)$ means $F'(x) = f(x)$.

Example 2. Find $\int x^2\ dx$

Example 3. Find $\int \frac{1}{1+x^2}\ dx$

Example 4. Complete the table of Indefinite Integrals without the aid of a book or notes. Assume that $F(x)$ is an antiderivative of $f(x)$, and $G(x)$ is an antiderivative of $g(x)$.

$$\int cf(x)\ dx = _____$$

$$\int [f(x) \pm g(x)]\ dx = _____$$

$$\int k\ dx = _____$$

$$\int x^n\ dx, n \neq -1 = _____$$

$$\int \frac{1}{x}\ dx = _____$$

$$\int e^x\ dx = _____$$

$$\int a^x\ dx, a > 0 = _____$$

$$\int \sin x\ dx = _____$$

$$\int \cos x\ dx = _____$$

$$\int \sec^2 x\ dx = _____$$

$$\int \csc^2 x\ dx = _____$$

$$\int \sec x \tan x\ dx = _____$$

$$\int \csc x \cot x\ dx = _____$$

$$\int \frac{1}{1+x^2}\ dx = _____$$

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = _____$$

Example 5. Find $\int (10x^4 - 2\sec^2 x)\ dx$

Example 6. Find $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$

Definition

An equation that involves the derivative(s) of a function is called a **differential equation**.

Example 7. Find f if $f'(x) = x(2-x)^2$.

Given additional conditions allows us to solve for particular solutions rather than families of solutions.

Example 8. Find f if $f'(x) = e^x + \dfrac{20}{1+x^2}$ and $f(0) = -2$.

Example 9. Find f if $f''(x) = 8x^3 + 5$ and $f(1) = 0$, $f'(1) = 8$.

Example 10. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find the position of the particle at t seconds.

3 Sums & Area

3.1 Sigma Notation

Definition

If $a_m, a_{m+1}, \ldots, a_n \in \mathbb{R}$ and $m, n \in \mathbb{Z}$ with $m \leq n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

We call i the **index** of the summation, m the **lower bound**, and n the **upper bound**. The left side of the definition is called **sigma notation**, and the right side is called **expanded form**.

Sigma notation is much like function notation, except we use a_i instead of $f(i)$ to represent the summand. Using function notation, it would look like the following.

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Example 1. Write the sum in expanded form: $\sum_{i=1}^{5} \sqrt{i}$.

Example 2. Write the sum in sigma notation.

$$1 + 4 + 9 + 16 + 25 + 36 + \cdots + 144$$

Example 3. Write the sum in expanded form: $\displaystyle\sum_{i=3}^{6} \frac{i^2(3i-1)}{i+1}$.

The following theorem will be incredibly helpful for evaluating much more complicated sums. This will actually be the focus of the next subsection. We will prove a couple of these.

Theorem

If $c \in \mathbb{R}$ and n is a positive integer, then

- $\displaystyle\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$
- $\displaystyle\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$
- $\displaystyle\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$
- $\displaystyle\sum_{i=1}^{n} 1 = n$
- $\displaystyle\sum_{i=1}^{n} c = nc$
- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Proof: We want to show that $\displaystyle\sum_{i=1}^{n} c = nc$.

Proof: We want to show that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Example 4. Write the sum without sigma. Simplify your result.

$$\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3-6\left(\frac{3i}{n}\right)\right]\left(\frac{3}{n}\right)$$

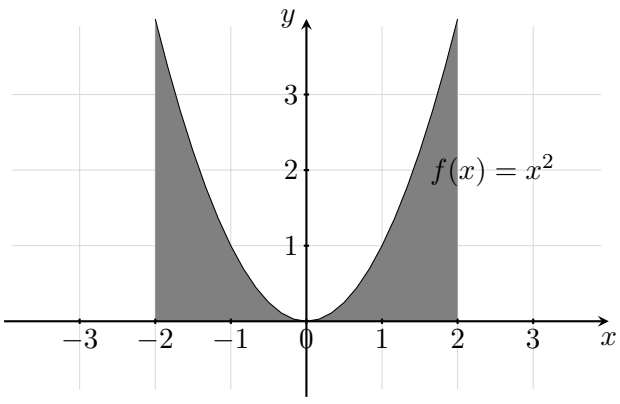
3.2 Areas

We know how to find the areas of specific geometric shapes. For example,

Example 5. Draw then write down a formula for the area of the following shapes.

- Triangle
- Square
- Rectangle
- Parallelogram
- Circle
- Semicircle

Now, what if we had the following shape? How would we find its area?



The strategy we will build next is an extension of what we did in differential calculus to produce a limit. We will find a crude approximation for the quantity we are trying to find (the area), then we will improve upon that approximation, then include a limit to find the exact value. The following example is used to mimic this process.

Ex: Let A represent the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$. Use rectangles to approximate A.

(a) Using 4 rectangles, show that $0.2188 < A$.

(b) Using 4 rectangles, show that $A < 0.4688$.

Using this same strategy, it can be shown that

n	L_n	R_n
4	$L_4 \approx 0.2188$	$R_4 \approx 0.4688$
8	$L_8 \approx 0.2734$	$R_8 \approx 0.3984$
1000	$L_{1000} \approx 0.3328335$	$R_{1000} \approx 0.3338335$

Based on the information above, what is an estimate for the exact value of A?

Definition

The **area** of the region S that lies under the graph of the continuous function f is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $x_i^* \in [x_{i-1}, x_i]$.

Example 6. Use limits and sigma notation to show that A in the previous example is exactly $\frac{1}{3}$.

4 Definite Integrals & Average Values

4.1 The Definite Integral

Definition

The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann Sum** and is an approximation for the area of a region under a curve.

Definition

If f is a function defined on $[a, b]$, we divide $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. Letting $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals and $x_1^*, x_2,^*, \ldots, x_n^*$ be **sample points** in these subintervals so that $x_i^* \in [x_{i-1}, x_i]$, we define the **definite integral of f from a to b** to be

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

In $\int_a^b f(x)dx$, we call $\int$ the **integral sign**, a the **lower limit**, b the **upper limit**, a and b together the **limits of integration**, $f(x)$ the **integrand**, and dx indicates that x is our variable.

When we first learn about numbers, they are presented as being positive. When negatives are introduced, we learn that, geometrically, the negative sign represents a direction. Similarly, distances are generally introduced as positive, but a negative distance generally means traveling backwards. Angles are also usually positive, but a signed angle indicates a direction of the opening.

At this point, we will look at area being signed. For area on a Cartesian plane, a positive area represents area that exists above the independent axis, and negative area is area that exists beneath the independent axis.

Definition

A definite integral can be interpreted as **net area**:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is area above the x-axis and A_2 below.

We will work through an integral in a few moments, but before doing so, we want to understand what functions are integrable as well as how to integrate.

Theorem

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$ and $\int_a^b f(x)\ dx$ exists.

In order to integrate, we will either interpret an integral as a signed area (geometrically), or work through the limit of a Riemann sum (algebraically). The simplest way to work through the limit of a Riemann sum is to choose sample points to be right endpoints; that is, $x_i^* = x_i$.

Definition

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Example 1. Express $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\Delta x$ as an integral on $[0, \pi]$.

Example 2. Evaluate $\int_0^3 (x^3 - 6x)\ dx$.

Example 3. Evaluate $\int_0^1 \sqrt{1-x^2}\ dx$

Example 4. Evaluate $\int_0^3 (x-1)\ dx$

Properties of Integrals

If $a, b, c \in \mathbb{R}$ and f is integrable on $[a, b]$, then

(a) $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

(b) $\int_a^a f(x)\ dx = 0$

(c) $\int_a^b c\ dx = c(b-a)$

(d) $\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

(e) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$

(f) $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

(g) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq 0$.

(h) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$.

(i) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$.

Example 5. Evaluate $\int_0^1 (4 + 3x^2)\ dx$.

4.2 Average Value

Definition

The **average value of** f on the interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\ dx$$

Example 6. Find the average value of the function $f(x) = x^3 - 6x$ on the interval $[0, 3]$.

Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists a number $c \in [a, b]$ such that

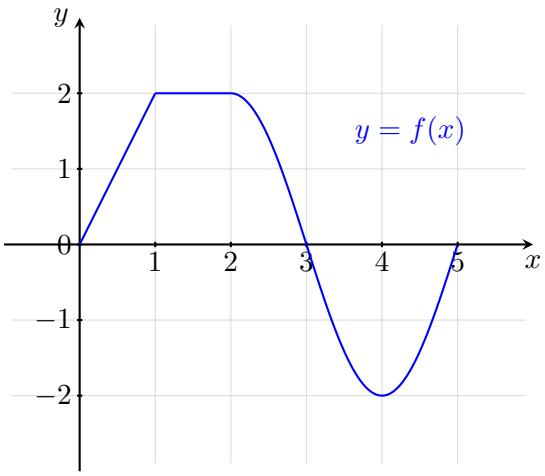
$$f(c) = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\ dx \qquad \text{so} \qquad \int_a^b f(x)\ dx = f(c)(b-a)$$

In Differential Calculus, there was a MVT for Derivatives. The MVT for Integrals is a consequence of that and the Fundamental Theorem of Calculus. This theorem is instrumental to the development of integral calculus.

5 The Fundamental Theorem of Calculus

5.1 The Accumulation Function

Example 1. Given the graph of f below and $g(x) = \int_0^x f(t)\,dt$, estimate $g(0), g(1), g(2), g(3), g(4)$, and $g(5)$. Sketch a rough graph of g.



$g(0) =$

$g(1) =$

$g(2) =$

$g(3) \approx$

$g(4) =$

$g(5) \approx$

5.2 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$, then $g(x) = \int_a^x f(t)\,dt$ where $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover, $g'(x) = f(x)$.

Example 2. Find $g'(x)$ if $g(x) = \int_0^x \sqrt{1 + t^2}\,dt$.

Example 3. Find $\frac{d}{dx} \int_{10}^x \sqrt{1 + t^2}\,dt$.

Example 4. Find the derivative of $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)\ dt$ (Fresnel function, used in optics)

Example 5. Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\ dt$

Example 6. Find the derivative of $h(x) = \displaystyle\int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\ dz$

5.3 FTC 2

The Fundamental Theorem of Calculus Part 2

If f is continuous on $[a, b]$, then $\int_a^b f(x)\ dx = F(b) - F(a)$, where $F' = f$.

Together, the FTC 1 and FTC 2 can be stated as such.

FTC

If f in continuous on $[a, b]$ and F is any antiderivative of f, then

1. $\dfrac{d}{dx}\int_a^x f(t)\ dt = f(x)$
2. $\int_a^b F'(x)\ dx = F(b) - F(a)$

Example 7. Evaluate $\int_1^3 e^x\ dx$.

Convention

$F(x)\Big|_a^b = F(b) - F(a)$

Example 8. Find the area under the parabola $y = x^2$ from 0 to 1.

Example 9. Evaluate $\int_0^3 (x^3 - 6x)\ dx$.

Example 10. Evaluate $\int_3^6 \frac{dx}{x}$.

Example 11. What is wrong with the following?

$$\int_{-1}^{3} \frac{1}{x^2}\, dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{3} = \frac{-1}{3} - 1 = -\frac{4}{3}$$

5.4 Net Change Theorem

Net Change Theorem

The integral of a rate of change is the net change: $\int_a^b F'(x)\, dx = F(b) - F(a)$.

Here are some practical examples of the Net Change Theorem.

- If the rate of growth of a population is $\frac{dn}{dt}$, then

  $$\int_{t_1}^{t_2} \frac{dn}{dt}\, dt = n(t_2) - n(t_1)$$

  is the net change in population from time t_1 to time t_2
- If an object moves in a straight line with position $s(t)$, then its velocity is $v(t)$, so

  $$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)$$

  is the net change of position (*displacement*) from time t_1 to time t_2
- If we want to calculate the *distance* during that time interval, then we integrate $|v(t)|$:

  $$\text{Total Distance} = \int_{t_1}^{t_2} |v(t)|\, dt$$

  Note: $\int_a^b |f(x)|\, dx \neq \left|\int_a^b f(x)\, dx\right|$
- The acceleration of an object is $a(t) = v'(t)$, so

  $$\int_{t_1}^{t_2} a(t)\, dt = v(t_2) - v(t_1)$$

  is the change in velocity from time t_1 to time t_2

Example 12. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ m/s. Find the displacement of the particle during the time period $1 \leq t \leq 4$, then find the total distance traveled during this time.

5.5 Proof of FTC

Proof of FTC1:

Proof. If $x, x+h \in (a,b)$, then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt \\ &= \left(\int_a^x f(t)\ dt + \int_x^{x+h} f(t)\ dt\right) - \int_a^x f(t)\ dt \\ &= \int_x^{x+h} f(t)\ dt \end{aligned}$$

Thus, for $h \neq 0$, $\dfrac{g(x+h)-g(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\ dt$.

Let's assume that $h > 0$. Since f is cont on $[x, x+h]$, the EVT says that there are numbers $u, v \in [x, x+h]$ such that $f(u) = m$ and $f(v) = M$ are the absolute max/min of f on $[x, x+h]$.

By Property 9 of Integrals, $mh \leq \displaystyle\int_x^{x+h} f(t)\ dt \leq Mh$, so $f(u)h \leq \displaystyle\int_x^{x+h} f(t)\ dt \leq f(v)h$.

Since $h > 0$, $f(u) \leq \frac{1}{h}\displaystyle\int_x^{x+h} f(t)\ dt \leq f(v)$, so

$$f(u) \leq \frac{g(x+h)-g(x)}{h} \leq f(v)$$

We could similarly work out if $h < 0$, but we will omit that.

As $h \to 0$, $u \to x$ and $v \to x$, since $x \leq u, v \leq x+h$. Therefore,

$$\lim_{h\to 0} f(u) = \lim_{u \to x} f(v) = f(x) \qquad \lim_{h\to 0} f(v) = \lim_{v \to x} f(v) = f(x)$$

By the Squeeze Theorem,

$$\lim_{h\to 0} f(u) \leq \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \leq \lim_{h\to 0} f(v) \Longrightarrow f(x) \leq g'(x) \leq f(x)$$

Therefore, $g'(x) = f(x)$. □

Proof of FTC2:

Proof. Let $g(x) = \int_a^x f(t)\ dt$. By FTC1, $g'(x) = f(x)$, so g is an antiderivative of f. If F is any other antiderivative of f, then $F(x) = g(x) + C$ on $[a,b]$ since F and g are continuous on $[a,b]$.

We claim that $\int_a^b f(x)\ dx = F(b) - F(a)$. If $x = a$, then $g(a) = \int_a^a f(t)\ dt = 0$, so

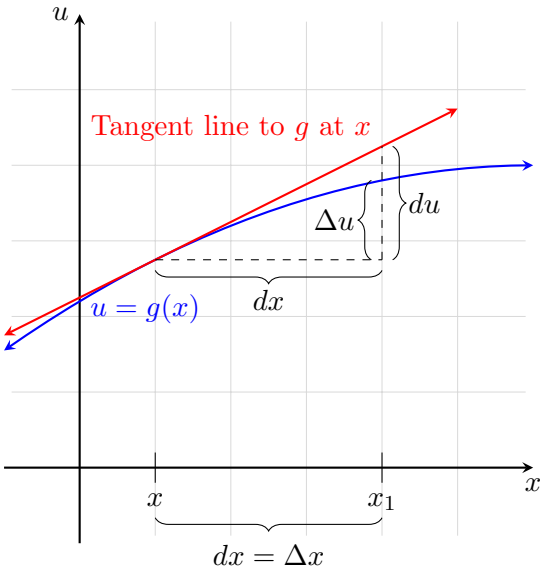
$$F(b) - F(a) = [g(b) + C] - [g(a) + C] = g(b) - g(a) = g(b) = \int_a^b f(t)\ dt$$

□

6 Substitution

6.1 Differentials

Differentials are complicated mathematical expressions that, up until now, have not been treated as a quantity. However, the differentials dx, dy, and others can be used to represent quantities, as well! The following is an image used to show what differentials represent.



In this graph, we have the graphs of $u = g(x)$ and the tangent line to u at x. We also see that the independent variable is changing from x to x_1 by an amount we generally understand as Δx. While x changes by Δx, we see that u changes by Δu, and we can see this along the blue graph of $u = g(x)$.

If we instead track the change in x along the red tangent line, then we will be observing an *approximate* change in u, and we call this quantity du. Our goal is to understand the relationship between du and x.

Considering dx and du as quantities, the slope of the red line at x is $\dfrac{du}{dx}$. The slope can also be computed from the curve $u = g(x)$ using a derivative to find $g'(x)$. Thus, $\dfrac{du}{dx} = g'(x)$. Multiplying by dx (because it is a quantity), we get the relationship we are looking for.

Differential Relationship

If $u = g(x)$ is differentiable, then $du = g'(x)\,dx$.

6.2 Substitution

We will be using differentials to establish the strategy of substitution. Here is an example.

Consider $\int 2x\sqrt{1+x^2}\ dx$. Suppose $u = 1 + x^2$. Then the differential relationship is $du = 2x\ dx$.

$\int 2x\sqrt{1+x^2}\ dx =$

Recall the **Chain Rule:** $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$.

If we substitute $u = g(x)$; then $du = g'(x)\ dx$, and

$$\begin{aligned}\int F'(g(x))g'(x)\ dx &= F(g(x)) + C \\ &= F(u) + C \\ &= \int F'(u)\ du\end{aligned}$$

This proves the Substitution Rule. We state it below using $f = F'$.

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))\ g'(x)\ dx = \int f(u)\ du$$

The purpose of Substitution is to take an integral that we may not know how to work out and replace it with something that we may know how to work out.

Example 1. Find $\int x^3\cos(x^4+2)\ dx$.

Example 2. Find $\int \sqrt{2x+1}\ dx$.

Example 3. Find $\int \frac{x}{\sqrt{1-4x^2}}\ dx$.

Example 4. Find $\int e^{252x}\ dx$.

Example 5. Find $\int \tan x \; dx$.

6.3 Substitution with Definite Integrals

Substitution with Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x)) \; g'(x) \; dx = \int_{g(a)}^{g(b)} f(u) \; du$$

Proof. Let F be an antiderivative of f. Then by FTC2

$$\int_a^b f(g(x)) \; g'(x) \; dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$$

On the other hand,

$$\int_{g(a)}^{g(b)} f(u) \; du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

□

Example 6. Evaluate $\int_0^4 \sqrt{2x+1} \; dx$.

Example 7. Evaluate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

Consider even/odd functions. – $\displaystyle\int_{-a}^{a} f(x)\,dx$.

6.4 Even and Odd Functions

Recall that a function is **even** if $f(-a) = f(a)$, and a function is **odd** if $f(-a) = -f(a)$.

Even Functions	**Odd Functions**
$f(-x) = f(x)$	$f(-x) = -f(x)$
Graph is symmetric about the y-axis	Graph is symmetric about the origin
Examples	Examples
c	x
x^2	x^3
$x^{-2} = \dfrac{1}{x^2}$	$x^{-3} = \dfrac{1}{x^3}$
x^4	x^5
$x^{-4} = \dfrac{1}{x^4}$	$x^{-5} = \dfrac{1}{x^5}$
x^{2n}	x^{2n+1}
$\cos x$	$\sin x$
$\sec x$	$\csc x$
$\lvert x\rvert$	$\tan x$
	$\cot x$
Product of Even Functions	Product of Even and Odd Function
Product of Odd Functions	
Sum of Even Functions	Sum of Odd Functions

Theorem

Suppose f is continuous on $[-a, a]$.

1. If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
2. If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Example 8. Evaluate $\int_{-252}^{252} \sin x \; dx$.

Example 9. Evaluate $\int_{-3}^{3} x^6 + 1$.

Example 10. Evaluate $\int_{-1}^{1} \dfrac{\tan x}{1 + x^2 + x^4}$.

Proof of Theorem: Split

$$\int_{-a}^{a} f(x) \; dx = \int_{-a}^{0} f(x) \; dx + \int_{0}^{a} f(x) \; dx = -\int_{0}^{-a} f(x) \; dx + \int_{0}^{a} f(x) \; dx$$

Let $u = -x$, so $du = -dx$, and when $x = -a$, $u = a$, so

$$-\int_{0}^{-a} f(x) \; dx = -\int_{0}^{a} f(-u) \; (-1) \; du = \int_{0}^{a} f(-u) \; du$$

Thus,

$$\int_{-a}^{a} f(x) \; dx = \int_{0}^{a} f(-u) \; du + \int_{0}^{a} f(x) \; dx$$

If f is even...

If f is odd... □

7 Integration By Parts

Recall the **Product Rule:** $\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + g(x)f'(x)$.

Then the following must be true.

$$\int \left(f(x)g'(x) + g(x)f'(x)\right)\,dx = f(x)g(x)$$
$$\int \left(f(x)g'(x)\right)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$
$$\int \left(f(x)g'(x)\right)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

This gives a formula for the integral of a product. Suppose $u = f(x)$ and $v = g(x)$. Then

$$u = f(x) \qquad dv = g'(x)\,dx$$
$$du = f'(x)\,dx \qquad v = g(x)$$

By Substitution, we have a theorem known as Integration by Parts.

Integration By Parts

If $u = f(x)$ and $v = g(x)$, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int \left(g(x)f'(x)\right)\,dx$$
$$\int u\,dv = uv - \int v\,du$$

Example 1. Find $\int te^{-3t}\,dt$.

Example 2. Evaluate $\int \ln x \ dx$.

Example 3. Evaluate $\int t^2 \sin 2t \ dt$.

Example 4. Evaluate $\int e^{2x} \cos x \; dx$.

8 Trigonometric Integrals

We will focus on powers of sine and cosine multiplied together. The general strategy is to use the Pythagorean Theorem to try to get just a single factor of $\sin x$ or $\cos x$, while the rest of the factors are the other function.

Strategy for Evaluating $\int \sin^m x \cos^n x \; dx$:

(a) If n is odd ($n = 2k + 1$), use the Pythagorean Theorem to convert cosines to sine, but save one cosine factor.

$$\int \sin^m x \cos^n x \; dx = \int \sin^m x (\cos^2 x)^k \cos x \; dx = \int \sin^m x (1 - \sin^2 x)^k \cos x \; dx$$

Then use substitution with $u = \sin x$ and $du = \cos x \; dx$.

(b) If m is odd ($n = 2k + 1$), use the Pythagorean Theorem to convert sines to cosine, but save one sine factor.

$$\int \sin^m x \cos^n x \; dx = \int \sin x (1 - \cos^2 x)^k \cos^n x \; dx = \int (1 - \cos^2 x)^k \cos^n x \sin x \; dx$$

Then use substitution with $u = \cos x$ and $-du = \sin x \; dx$.

(c) If m and n are both even, use half-angle identities:

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

and sometimes $\sin x \cos x = \frac{1}{2} \sin 2x$

Example 1. Evaluate $\int \cos^3 x \; dx$.

Example 2. Evaluate $\int \sin^2\theta \cos^5\theta \; d\theta$.

Example 3. Evaluate $\int_0^{\pi} \cos^4 t \; dt$.

9 Partial Fraction Decomposition

Notice

$$\frac{5}{x-1} - \frac{2}{x-2} = \frac{5(x-2)-2(x-1)}{(x-1)(x-2)} = \frac{3x-8}{x^2-3x+2}$$

So

$$\int \frac{3x-8}{x^2-3x+2}\,dx = \int \left(\frac{5}{x-1} - \frac{2}{x-2}\right)\,dx$$

Partial fraction decomposition is a process of rewriting a single fraction as a sum of fractions with simpler denominators called "partial fractions". The partial fractions can then be integrated using substitution or with a known antiderivative.

In order to use the process of partial fraction decomposition, we will

1. Factor the denominator.
2. For each distinct linear factor in the denominator, write a term of the form $\frac{A}{\text{factor}}$.
3. Find each unknown constant in the numerator using any of
   - A system of equations,
   - The wipeout method, or
   - The Heaviside Coverup.
4. Integrate the partial fraction decomposition.

Example 1. Evaluate $\displaystyle\int \frac{3x-8}{x^2-3x+2}\,dx$.

Example 2. Evaluate $\int \frac{x}{x^2-1}\,dx$.

Example 3. Evaluate $\int_0^1 \frac{2x-1}{6x^2-5x-6}\,dx$.

10 Numerical Integration

10.1 Summary of Formulas for this Section

$$
\begin{aligned}
L_n &= \Delta x(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})) \\
R_n &= \Delta x(f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)) \\
M_n &= \Delta x\left(f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right) \\
T_n &= \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)) \\
S_n &= \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))
\end{aligned}
$$

10.2 Rectangle and Trapezoid Approximations

Sometimes definite integration may seem impossible to calculate, particularly because FTC requires us to find an antiderivative that we may not be able to find. Other times we may not even have a formula (experiments). In each of these cases, we would need an approximation strategy, and our go-to is to use a Riemann sum.

We have used rectangles previously, and so we've seen the following.

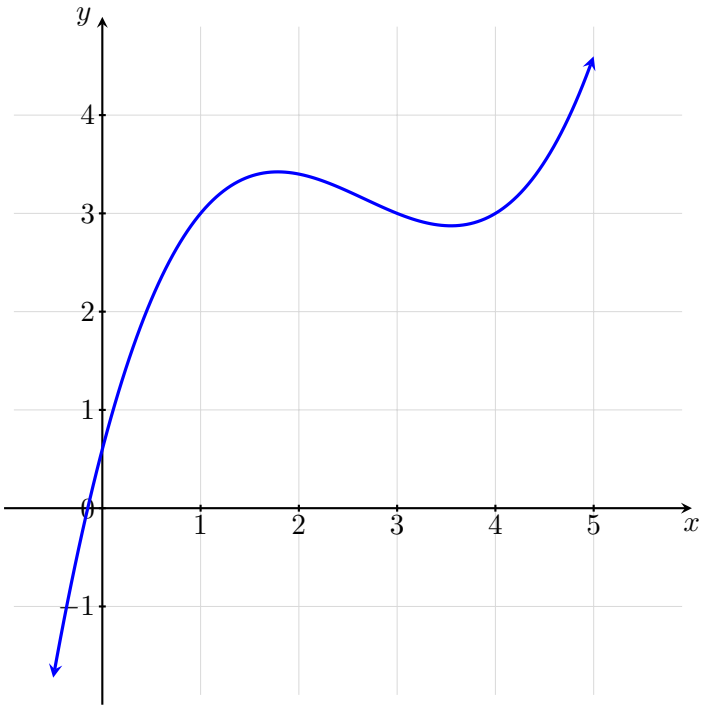
$$
\begin{aligned}
\int_a^b f(x)\,dx &\approx \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&\approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \\
&\approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x
\end{aligned}
$$

The approximations L_n and R_n are called **left and right endpoint approximations**.

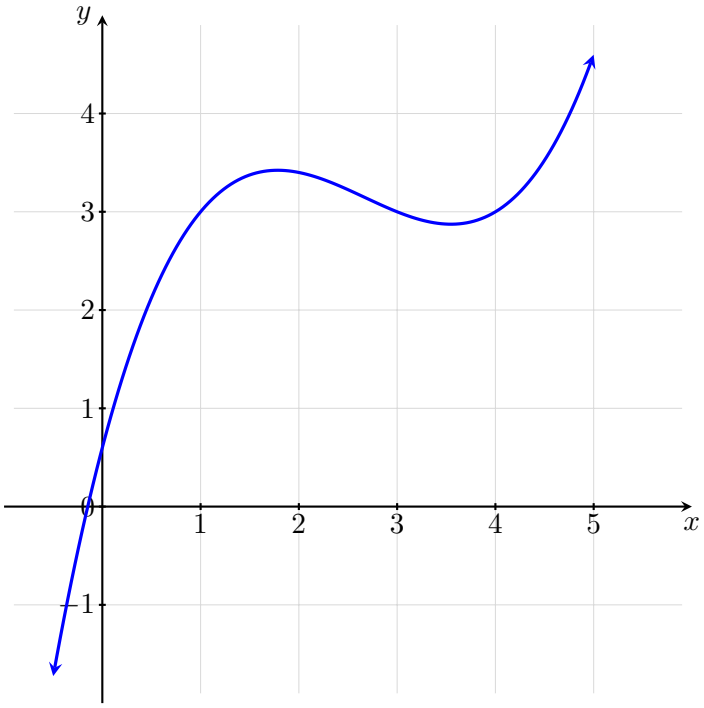
If we choose $x_i^* = \overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$, then we obtain what is called the Midpoint Rule.

10.2.1 Geometric Justifications of the Midpoint and Trapezoid Rules

Midpoint Rule: $\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\overline{x_i})\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$.



Trapezoid Rule: $\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n))$ where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.



Example 1. Approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$ using T_6 and M_6.

10.3 Error

Defn: The **error** in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact.

In particular,

$$\int_a^b f(x)\ dx = \text{approximation} + \text{error}$$

Thus

$$E_T \approx -0.002488 \qquad E_M \approx 0.001239$$

And in general

$$E_T = \int_a^b f(x)\ dx - T_n \qquad E_M = \int_a^b f(x)\ dx - M_n$$

Observations:

- Approximations improve as n increases.
- Careful of accumulated roundoff error!
- E_L and E_R have opposite sign.
- E_L and E_R are about halved as n doubles.
- T_n and M_n are much more accurate than R_n and L_n.
- E_T and E_M are about quartered as n doubles.
- $|E_M| \approx \frac{1}{2}|E_T|$.

Error Bounds

Suppose $|f''(x)| \le K$ for $a \le x \le b$. Then $|E_T| \le \dfrac{K(b-a)^3}{12n^2}$ and $|E_M| \le \dfrac{K(b-a)^3}{24n^2}$.

From our example, if $f(x) = \dfrac{1}{x}$, then $f''(x) = \dfrac{2}{x^3}$, and since $1 \le x \le 2$,

$$|f''(x)| = |\frac{2}{x^3}| \le \frac{2}{1^3} = 2$$

So $K = 2, a = 1, b = 2, n = 5$,

$$E_T \le \frac{2(2-1)^3}{12(5)^2} = \frac{1}{150} \approx 0.006667$$

Example 2. How large should n be to guarantee that the T_n and M_n approximations for $\int_1^2 \frac{1}{x}\,dx$ are accurate within 0.0001?

10.4 Simpson's Rule

Simpson's Rule is another approximation strategy for definite integrals. This rule is named after Thomas Simpson (1710 - 1761), who did not come up with this rule but rather popularized it. This rule uses parabolas rather than polygons to establish the formula. This is beneficial because most curves are better approximated by parabolas than with straight-line segments. This rule is also super helpful because it takes just as much work to use T_n as it is to use S_n.

A proof of Simpson's Rule is provided at the end of the section.

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n))$$

where n is even and $\Delta x = \frac{b-a}{n}$.

Example 3. Approximate $\int_1^2 \frac{1}{x}\,dx$ with S_6.

It turns out, $S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$.

10.5 Establishing Simpson's Rule

Divide $[a, b]$ into an even number of subintervals, n, of equal length $h = \Delta x = \frac{b-a}{n}$.

On each consecutive pair of intervals, approximate $y = f(x) \geq 0$ by a parabola.

If $y_i = f(x_i)$ and P_i is the point $P_i(x_i, y_i)$, then a typical parabola passes through P_i, P_{i+1}, P_{i+2}.

Suppose $x_0 = -h, x_1 = 0, x_2 = h$. The parabola through P_1, P_2, P_3 is $y = Ax^2 + Bx + C$, so

$$\int_{-h}^{h} (Ax^2 + Bx + C)\ dx = 2\int_{0}^{h} (Ax^2 + C)\ dx = 2\left[A\frac{x^3}{3} + Cx\right]_0^h = \frac{h}{3}(2Ah^2 + 6C)$$

Since the parabola passes through $P_1(-h, y_0), P_2(0, y_1), P_3(h, y_2)$,

$$\begin{aligned} y_0 &= A(-h)^2 + B(-h) + C = Ah^2 - Bh + C \\ y_1 &= C \\ y_2 &= Ah^2 + Bh + C \end{aligned}$$

And $2Ah^2 + 6C = y_0 + 4y_1 + y_2$. Thus,

$$\int_{-h}^{h} = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Horizontally shifting a parabola does not change the area under the curve. Thus, the area under the parabola through P_0, P_2, P_2 from $x = x_0$ to $x = x_2$ is still $\frac{h}{3}(y_0 + 4y_1 + y_2)$. Moreover, the area under the parabola through P_2, P_3, P_4 is $\frac{h}{3}(y_2 + 4y_3 + y_4)$.

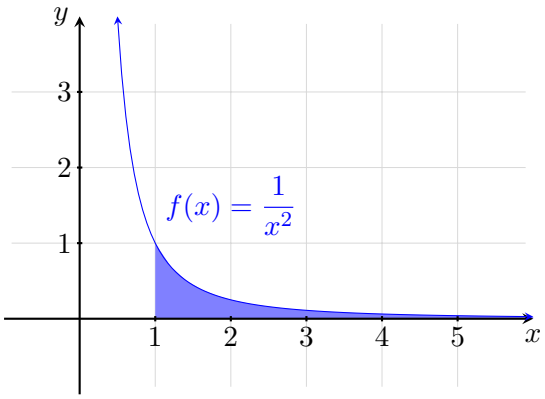
Thus,

$$\begin{aligned} \int_a^b f(x)\ dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \end{aligned}$$

11 Improper Integrals

An **improper integral** is essentially a definite integral either over an infinite interval or with an infinite discontinuity.

Consider the region S under $y = \dfrac{1}{x^2}$ right of $x = 1$ that stretches an infinite length to the right. How big is S?



Feel free to use this applet to aid in the following.

Suppose S is between $x = 1$ and $x = t$. Then the area of S is

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{t}\bigg|_1^t = 1 - \frac{1}{t}$$

So $A(t) < 1$ for all $t > 1$. Notice

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

Thus, $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$

Definition

If $\displaystyle\int_a^t f(x)\,dx$ exists for all $t \geq a$, then we define $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$, provided the limit exists (as a finite number). Moreover, if $\displaystyle\int_t^b f(x)\,dx$ exists for all $t \leq b$, then $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$, provided the limit exists.

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists. If the limit DNE, we call them **divergent**.

11.1 Type 1 Improper Integrals

Definition

If $\int_a^\infty f(x)\ dx$ and $\int_{-\infty}^b f(x)\ dx$ are both convergent, then

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{a} f(x)\ dx + \int_a^{\infty} f(x)\ dx$$

for any $a \in \mathbb{R}$. Each of these last three integrals are called **Type 1 Improper Integrals**. If f is positive and $\int_a^\infty f(x)\ dx$ is convergent, then we define the area of S, the region under the curve $y = f(x)$ above the x-axis to be

$$A(S) = \int_a^{\infty} f(x)\ dx$$

Example 1. Determine whether $\displaystyle\int_a^\infty \frac{1}{x}\ dx$ is convergent or divergent.

Example 2. Evaluate $\displaystyle\int_{-\infty}^{0} xe^x\ dx$.

Example 3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Theorem

The improper integral $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ converges when $p > 1$ and diverges when $p \leq 1$.

Proof. We know $p = 1 \Rightarrow$ divergent. Assume $p \neq 1$. Then

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{1}{-p+1} x^{-p+1}\Big|_1^t = \frac{1}{1-p} \lim_{t\to\infty} \left(\frac{1}{t^{p-1}} - 1\right)$$

If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $\frac{1}{t^{p-1}} \to 0$, so $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1}$.

If $p < 1$, then $p - 1 < 0$, so as $t \to \infty$, $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$, so $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ diverges. □

11.2 Type 2 Improper Integrals

Suppose f is a positive continuous function defined on $[a, b)$ with a vertical asymptotes at b. If $t \in [a, b)$, then the area between $x = a$, $x = t$, $y = 0$, and $y = f(x)$ is $A(t) = \int_a^t f(x)\,dx$. Feel free to use this applet to explore this.

If $A(t) \to L$ as $t \to b^-$, then we define

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

We use this to define an improper integral even when f is not positive and no matter what kind of discontinuity f has at b.

Definition

If f is continuous on $[a,b)$ and discontinuous at b, then $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$ if the limit exists. Moreover, if f is continuous on $(a,b]$ and discontinuous at a, then $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$ if the limit exists.
The improper integral $\int_a^b f(x)\,dx$ is **convergent** if the corresponding limit exists and **divergent** if it does not.
If f has a discontinuity at $c \in (a,b)$ and $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. Each of these last three improper integrals are called **Type 2 Improper Integrals**.

Example 4. Determine whether the integral converges or diverges. If it converges, find its limit.

$$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$$

Example 5. Determine whether the integral converges or diverges. If it converges, find its limit.

$$\int_0^{\frac{\pi}{2}} \sec\theta\,d\theta$$

Example 6. Determine whether the integral converges or diverges. If it converges, find its limit.

$$\int_0^5 \frac{w}{w-2}\, dw$$

Example 7. Determine whether $\int_0^1 \ln y \, dy$ is convergent or divergent. If it converges, then evaluate the integral.

11.3 The Comparison Theorem

Comparison Theorem

Suppose f, g are continuous with $0 \leq g(x) \leq f(x)$ for $x \geq a$.

- If $\int_a^\infty f(x)\ dx$ is convergent, then $\int_a^\infty g(x)\ dx$ is convergent.
- If $\int_a^\infty g(x)\ dx$ is divergent, then $\int_a^\infty f(x)\ dx$ is divergent.

Example 8. Show $\displaystyle\int_0^\infty e^{-x^2}\ dx$ is convergent.

Solution: Note that $\int_0^\infty e^{-x^2}\ dx = \int_0^1 e^{-x^2}\ dx + \int_1^\infty e^{-x^2}\ dx$. FTC2 fails because the antiderivative is non-elementary.

Now, $\int_0^1 e^{-x^2}\ dx = A$, where A is some number. Moreover, when $x \geq 1$, $\ldots e^{-x^2} \leq e^{-x}$.

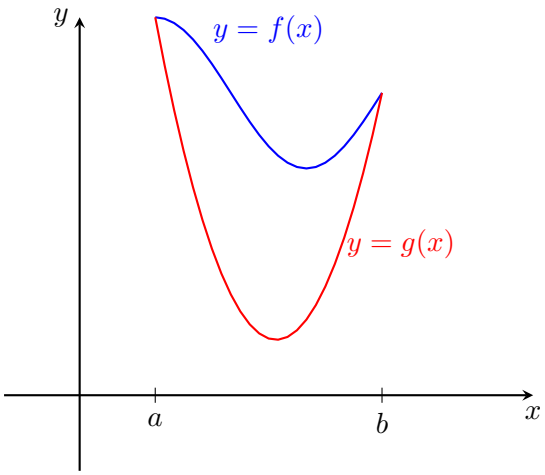
$$\int_1^\infty e^{-x}\ dx = \lim_{t\to\infty} \int_1^t e^{-x}\ dx = \lim_{t\to\infty} (e^{-1} - e^{-t}) = e^{-1}$$

By the Comparison Test, $\int_1^\infty e^{-x^2}\ dx$ converges, so $\int_0^\infty e^{-x^2}\ dx$ converges. □

12 Area Between Curves

12.1 Building the Definite Integral

Let S be the region between the two curves $f(x)$ and $g(x)$ between $x = a$ and $x = b$.



Just as before, we can subdivide the interval $[a, b]$ into n strips of equal width, $\Delta x = \frac{b-a}{n}$. We can then draw rectangles in each subinterval of width Δx. The height of each rectangle can be found by $f(x_i^*) - g(x_i^*)$. If A is the area of S, then

$$\begin{aligned} A &\approx \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x \\ &= \int_a^b (f(x) - g(x))\, dx \end{aligned}$$

12.2 Area Between Curves

Area Between Curves

The area A of the region bounded by the four curves $y = f(x)$ and $y = g(x)$, $x = a$ and $x = b$ where f and g are continuous and $f \geq g$ for all $x \in [a, b]$ is

$$A = \int_a^b (f(x) - g(x))\, dx$$

Example 1. Let S be the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$. Let A be the area of S.

(a) Sketch the graphs of $y = e^x$, $y = x$, and S on a Cartesian plane.

(b) Write an integral that represents A.

(c) Find the exact value of A.

Example 2. Let S be the region enclosed by $y = x^2$ and $y = 2x - x^2$. Let A be the area of S.

(a) Sketch the graphs of $y = x^2$, $y = 2x - x^2$, and S on a Cartesian plane.

(b) Write an integral that represents A.

(c) Find the exact value of A.

Sometimes $f(x) > g(x)$, and other times, $g(x) \geq f(x)$. We may need to use several integrals, but since

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x) \\ g(x) - f(x) & \text{when } g(x) \geq f(x) \end{cases}$$

we can write a single expression for area.

Definition

The area of the region enclosed by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$ is $A = \int_a^b |f(x) - g(x)|\, dx$.

Example 3. Find the area of the region bounded by the curves $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$.

(a) Sketch the graphs of $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \pi$ on a Cartesian plane.

(b) Write an integral that represents the area of the region.

(c) Find the exact value of the area described.

12.3 Horizontally Simple Regions

If a region is bounded by $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous with $f(y) \geq g(y)$ on $c \leq y \leq d$, then its area is

$$A = \int_c^d (f(y) - g(y))\, dy$$

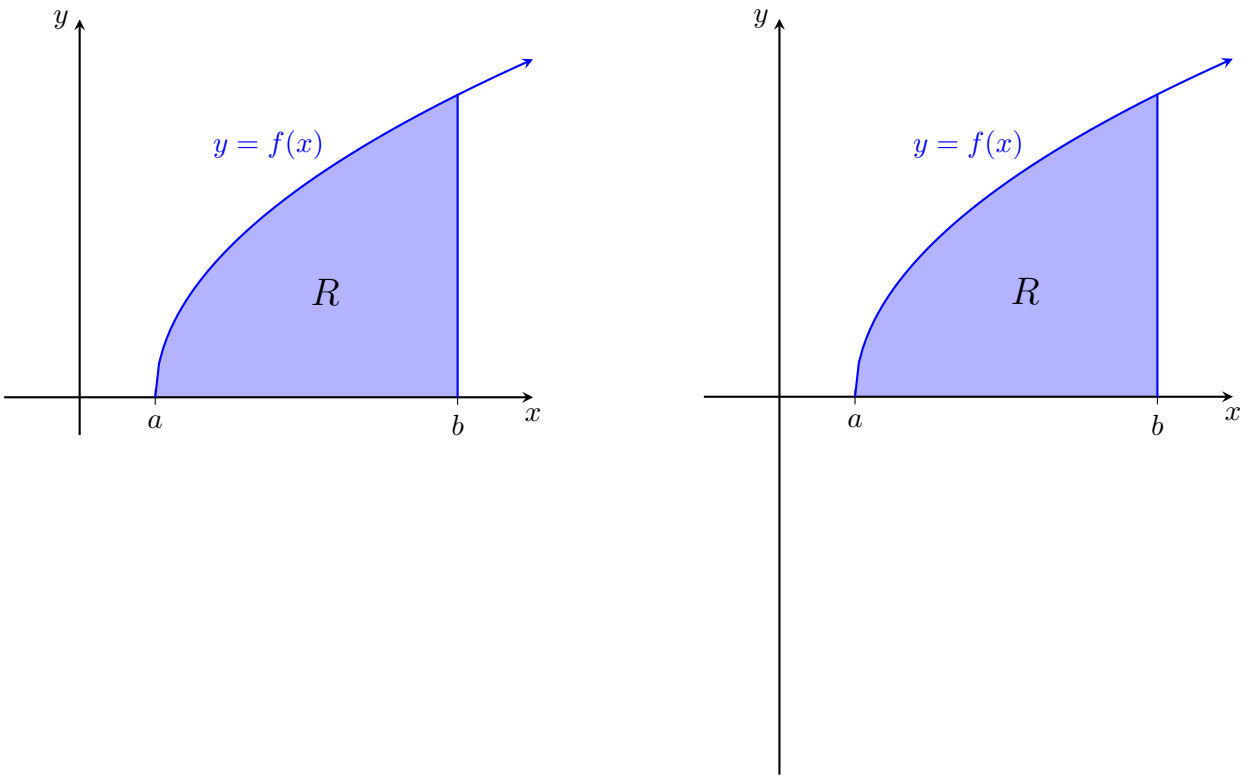
If x_R is the right boundary and x_L is the left boundary, then $A = \int_c^d (x_R - x_L)\, dy$.

Example 4. Find the area of the region enclosed by $y = x - 1$ and $y^2 = 2x + 6$.

13 Volumes

13.1 Volume by Disk

Let $y = f(x)$ and $a \leq x \leq b$. Let R be the region above the x-axis, beneath $y = f(x)$, to the right of $x = a$, and to the left of $x = b$.



Suppose we revolve R about the x-axis to produce the **solid of revolution** S. Our goal is to find the volume of S.

In order to accomplish this, we will proceed similar to how we have for the other integrals. Begin by subdividing the interval $[a, b]$ into n equal parts of width $\Delta x = \frac{b-a}{n}$. Name each of the subintervals $x_0, x_1, \ldots, x_n$, and select a sample point in each subinterval, x_i^*.

We want the volume of S, and S is shaped by taking R and revolving it about the x-axis. To approximate this, we will draw approximating rectangles as if we were finding the area of R. We will then revolve the rectangles about the x-axis to produce **disks**.

Thus, the volume of a disk is found to be $\pi R^2 \Delta x$. Then if V_i is the volume of the ith disk, then $V_i = \pi R(x_i^*)^2 \Delta x$. Then

$$\begin{aligned} V &\approx \sum_{i=1}^{n} V_i \\ &\approx \sum_{i=1}^{n} \pi R(x_i^*)^2 \Delta x \\ V &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi R_i(x)^2 \Delta x \\ &= \int_a^b \pi R(x)^2 \, dx \end{aligned}$$

The Disk Method

Suppose f is continuous on $[a, b]$, and R is the region in the Cartesian plane enclosed by the x-axis, $y = f(x)$, $x = a$, and $x = b$. Let S be the solid obtained by revolving R about the x-axis. Then the **volume**, V, of S is

$$V = \int_a^b \pi f(x)^2 \, dx$$

Definition

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \int_a^b A(x) \, dx$$

Example 1. Show the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Example 2. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

13.2 Volume by Washer

Suppose R is a Cartesian region, and we revolve R about an axis. If the cross section is not a disk but rather a washer, then our disk method does not work. We will omit the exploration of the formula for a volume by washers and rather present the formula.

The Disk Method

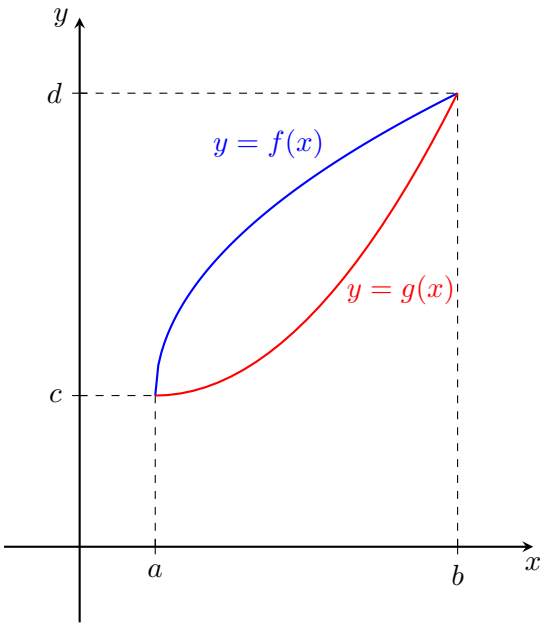
Suppose S is a solid that lies between $x = a$ and $x = b$ where the cross-sections are washers. Then the **volume**, V, of S is

$$V = \int_a^b \pi \left[R_{outer}(x)^2 - R_{inner}(x)^2 \right] \, dx$$

Example 3. Find the volume of the solid obtained by rotating the region enclosed by $y = x$ and $y = x^2$ about the line $y = 2$.

13.3 Summary of Formulas for this Section

The following are formulas for the volume of a solid of revolution. In each of these, R represents a function whose output is the radius of a typical shape from the center of revolution to the curve. The interval $[a, b]$ represents the smallest and largest x-values of the original region. The interval $[c, d]$ represents the smallest and largest y-values of the original region. For the shell method, h represents the height of a typical shell.



Disk Method	Washer Method	Shell Method
$V = \int_a^b \pi R(x)^2 \, dx$	$V = \int_a^b \pi \left[R_{out}(x) - R_{in}(x)\right]^2 dx$	$V = \int_a^b 2\pi R(x)h(x) \, dx$
$= \int_c^d \pi R(y)^2 \, dy$	$= \int_c^d \pi \left[R_{out}(y) - R_{in}(y)\right]^2 dy$	$= \int_c^d 2\pi R(y)h(y) \, dy$
Typical Disk	Typical Washer	Typical Shell

13.4 Volume by Cylindrical Shells

Suppose we are finding the volume of a solid of revolution, and we begin using the washer method. If we find that the outer and inner radii come from the same formula, then the washer method is not ideal to use. Instead, we use a method called the **Method of Cylindrical Shells**.

The Method of Cylindrical Shells

The volume of the solid obtained by rotating about the y-axis the region bounded by $y = f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, $a \leq b$ is

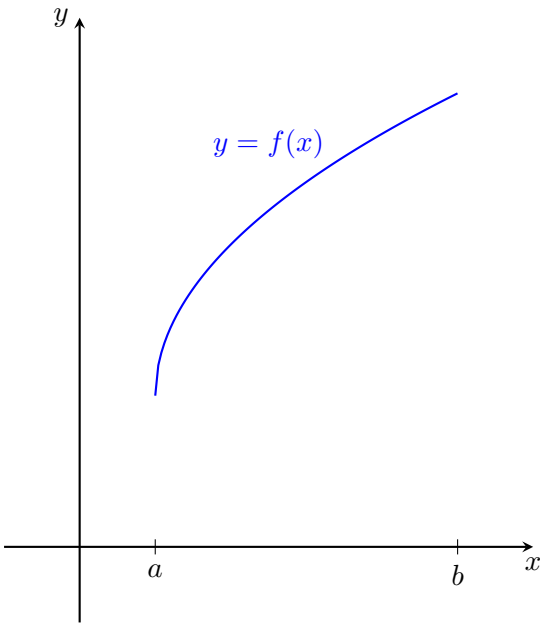
$$V = \int_a^b 2\pi x f(x)\, dx$$

Example 4. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Example 5. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

14 Arc Length

Suppose C is the curve $y = f(x)$, f is continuous, $x \in [a, b]$.



Divide $[a, b]$ into n equal subintervals length Δx with endpoints $x_0, x_1, \ldots, x_n$. If $y_i = f(x_i)$, then $P_i(x_i, y_i)$ lies on C and we can make a polygon with vertices $P_0, P_1, \ldots, P_n$ to approximate C.

The length L of C is approximately the length of the polygon and improves as n increases. Thus, the **length** L of the curve C with equation $y = f(x)$, $x \in [a, b]$ is found to be as $L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$.

If $\Delta y_i = y_i - y_{i-1}$, then by the distance formula, $|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Using the Mean Value Theorem, there is a number $x_i^* \in [x_{i-1}, x_i]$ such that $f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$. So

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\sqrt{(\Delta x)^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\Delta x \end{aligned}$$

So then we can define the length L as follows.

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \end{aligned}$$

Definition

If f' is continuous on $[a,b]$, then the **length** of the curve $y = f(x)$, where $x \in [a,b]$, is $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$.

In the event $x = g(y), y \in [c,d]$ and g' is continuous, then

$$L = \int_c^d \sqrt{1+[g'(y)]^2}\,dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Example 1. Write an integral that represents the length of the arc of the parabola $y^2 = x$ from the origin to $(1,-1)$.

Example 2. Consider the hyperbola $xy = 1$.

(a) Set up an integral for arc length of $xy = 1$ from $(1,1)$ to $\left(2, \frac{1}{2}\right)$.

(b) Estimate L using S_6

Definition

If C is a smooth curve with equation $y = f(x)$, $x \in [a, b]$, let $s(x)$ be the distance along C from $P_0(a, f(a))$ to $(x, f(x))$. Then $s(x)$ is called the **arc length function**, and $s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}\, dt$.

Moreover, since s is continuous, by FTC1, $\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, and finding the differential of arc length, $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$, and so $(ds)^2 = (dx)^2 + (dy)^2$.

So, if $L = \int ds$, then we can manipulate $(ds)^2 = (dx)^2 + (dy)^2$ to get $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ or $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$.

15 Work

15.1 Introduction to Work

In English – "work" means the total amount of effort required to do something. In Physics – "work" depends upon "force", which is akin to a push or pull.

Definition

If an object moves along a straight path with position function $s(t)$, then the **force** F on the object in the same direction is

$$F = ma = m\frac{d^2s}{dt^2} \qquad \text{(Newton's Second Law of Motion)}$$

where m is its mass.

As for units, we have two different systems. In metric: mass is kg, displacement is m, time is s, so F is measured in Newtons: $N = \text{kg} \cdot \text{m/s}^2$. In the Imperial system, force is measured in lbs.

We will use the following definitions to build a definite integral that represents the work done by a force.

Definition

If acceleration is constant, the force F is also constant, and the **work** done, W, is defined as

$$W = Fd$$

where d is the distance the object moves. If F is measured in newtons and d is in meters, then the unit for W is a Newton-meter, called a **joule**. If F is measured in pounds and d in feet, then the unit for W is a ft-lb, about 1.36 J.

Example 1. How much work is required to lift a 20-lb weight 6 feet off the ground?

This exploration assumes that force is constant. What if it varies?

15.2 Building a Definite Integral for Work

Suppose an object moves along the x-axis in the positive direction from $x = a$ to $x = b$, and at each point x between a and b, a force $f(x)$ acts on the object, where f is continuous. If we divide $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ of equal width Δx, and choose sample points $x_i^* \in [x_{i-1}, x_i]$, then the force at that point is $f(x_i^*)$. If n is quite large, then Δx is quite small, and since f is continuous, f doesn't change very much on $[x_{i-1}, x_i]$, so f is almost constant on the interval. Thus, the work W_i that is done in moving the particle from x_{i-1} to x_i is approximately

$$W_i \approx f(x_i^*)\Delta x, \qquad \text{so} \qquad W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Definition

We define the **work** done in moving an object from a to b to be

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\ dx$$

where $f(x)$ is the force acting on the object at $x \in [a, b]$.

Example 2. A particle is x feet from the origin. A force of $x^2 + 2x$ pounds acts on it. How much work is done in moving the particle from $x = 1$ to $x = 3$?

15.3 Springs

A common computation to work through is the work required to stretch a spring. In order to find this, we will need to use Hooke's Law.

Hooke's Law states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x. That is, $f(x) = kx$, where f is force and $k > 0$ is called the **spring constant**. Hooke's Law holds as long as x is not too large.

Example 3. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm? *Hint: Be careful of units.*

16 Hydrostatic Force

Hydrostatic pressure and force are the pressure and force acting on an object in a liquid.

Given a particle in liquid, hydrostatic force is simple. The force exerted on an object in a liquid is the force due to the liquid above it.

Consider a horizontal plate with area A m^2 submerged in a fluid with density $\rho \frac{\text{kg}}{\text{m}^3}$ at a depth of d meters. Then the fluid above the plate has volume $V = Ad$, so its mass is

$$m = \rho V = \rho A d.$$

This means that the force acting on the plate is

$$F = mg = \rho g A d$$

where g is the gravitational constant.

Definition

The **pressure** P on the plate is the amount of force per unit area – that is, $P = \dfrac{F}{A} = \rho g d$.
The SI unit for pressure is a **pascal**, $P = \frac{\text{N}}{\text{m}^2}$. Often, kilopascals are use (kPa).
To simplify, we often use $\delta = \rho g$ to represent **weight density** of a fluid (whereas ρ represents mass density). For water, $\rho = 1000 \frac{\text{kg}}{\text{m}^3}$, and $\delta = 62.5 \frac{\text{lb}}{\text{ft}^3}$.

Principle of Fluid Pressure

At any point in a liquid, the pressure is the same in all directions. The pressure in any direction at a depth d in a fluid with mass density ρ is

$$P = \rho g d = \delta d.$$

This means that the hydrostatic force on a horizonatl plate with area A and depth d submerged in a fluid with weight density δ is $F = PA = \delta d A$.

Example 1. Find the pressure at the bottom of a 2m deep swimming pool. Express your conclusion in kPa.

Example 2. A dam has the shape of a trapezoid with upper base 50 m, lower base 30 m, and height 20 m. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam. Round your conclusion to the nearest Newton.

Example 3. A cylindrical drum with radius 3 ft is submerged in water 10 feet deep. Find the hydrostatic force on one end of the cylindrical drum. Round your conclusion to the nearest pound.