MTH 251 Notes

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1 Precalculus Review

There are several concepts from precalculus that will be necessary for success in this course.

- The Laws of Exponents
- Expanding a Binomial Power
- Factoring
- Equation of a Line
- Simplifying Complex Rational Expressions
- Rationalizing a Denominator
- Unit Circle

The following are several examples relating to each of these concepts.

Example 1. Simplify $\left(\dfrac{3x^{-4}y^2}{-12x^0y}\right)^4$

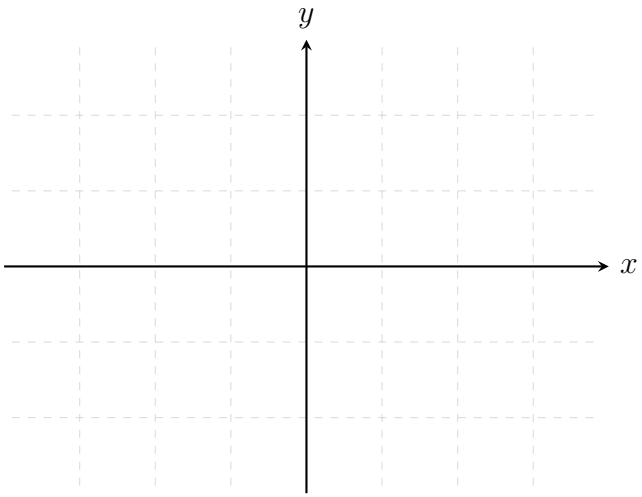
Example 2. Simplify $(4-xy)^2$

Example 3. Find an equation for the line through the points $(2, 5)$ and $(-1, 6)$.

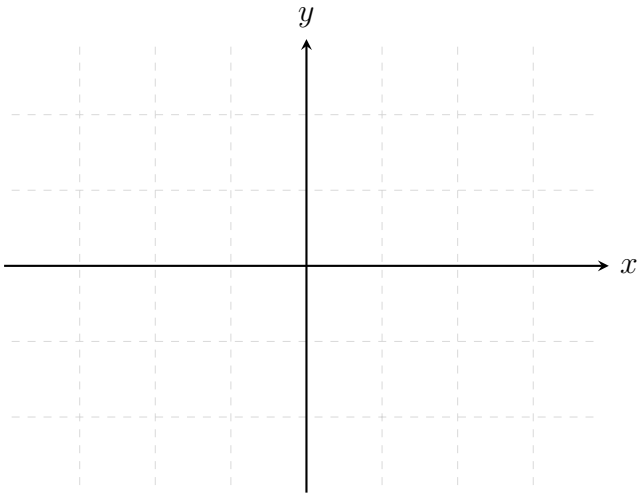
Example 4. Simplify $\dfrac{\dfrac{2a}{a+1} - \dfrac{4a+4}{a}}{\dfrac{a}{a^2} + \dfrac{1}{a^2+a}}$

Example 5. Sketch the graph of each function. Be sure to provide a scale on each set of axes.

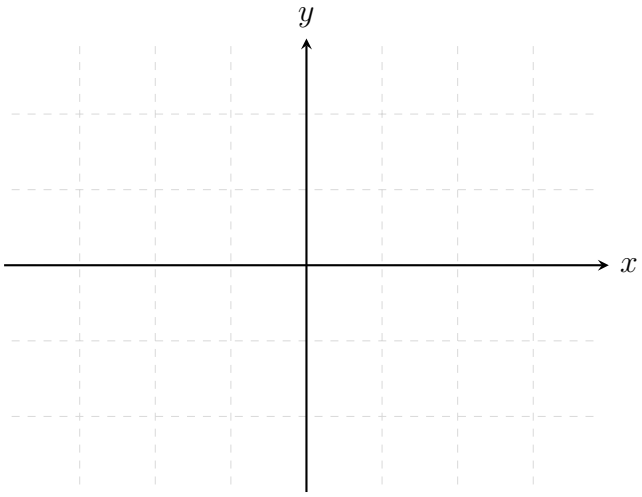
(a) $\sin x$



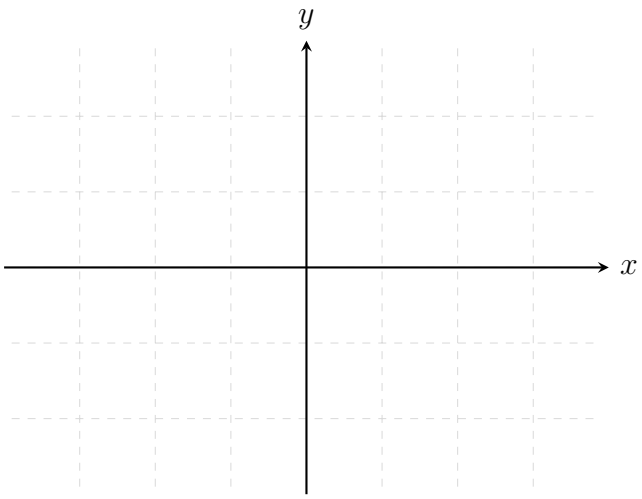
(b) $\tan x$



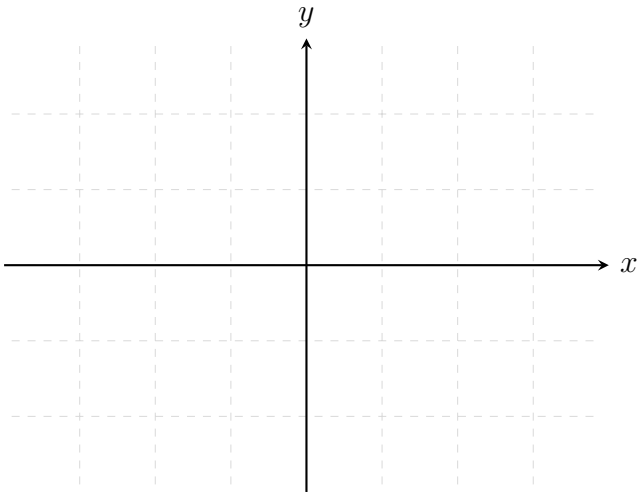
(c) $\ln x$



(d) e^x



(e) $\arctan x$



Example 6. Evaluate $\sin 120°$, $\cos \dfrac{11\pi}{6}$, $\tan \dfrac{8\pi}{3}$

2 Introduction to Calculus

We have two goals in this section.

- Find an equation of the line tangent to $y = x^2$ at $P(1, 1)$.
- Find the instantaneous velocity of a ball dropped from a high spot.

2.1 Tangents

Example 1. Find an equation of the line tangent to the graph of $y = x^2$ at the point $P(1, 1)$.

2.2 Velocity

Example 2. Suppose a ball is dropped from an observation deck of the CN Tower in Toronto at 450m above the ground. What is the velocity of the ball after 5 seconds? *Hint: The displacement of an object in free fall is given by* $s(t) = 4.9t^2$.

Definition

The **instantaneous velocity when** $t = a$ is defined as the limiting value of the average velocities in smaller and smaller intervals ending with $t = a$.

3 Limits

Consider the function $f(x) = x^2 - x + 2$. Let's look around *but not at* $x = 2$.

x	$f(x)$		x	$f(x)$
1	2		3	8
1.5	2.75		2.5	5.75
1.9	3.71		2.1	4.31
1.99	3.9701		2.01	4.0301
1.999	3.997001		2.001	4.003001
1.99999	3.9999700001		2.00001	4.0000300001

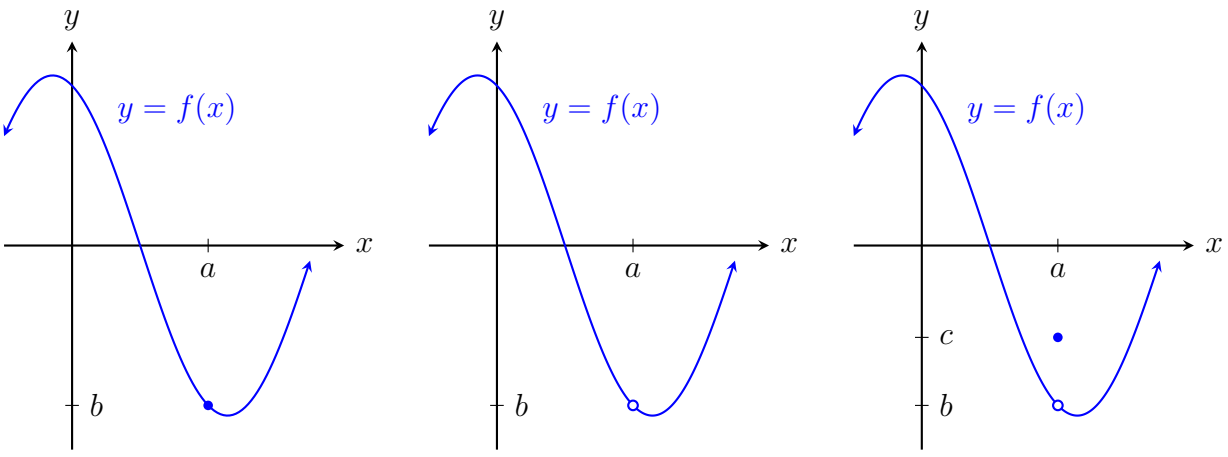
Use Desmos to look at the graph of $y = f(x)$. Though it seems obvious, it looks like as x approaches 2, the values of $f(x)$ consequently approach 4. On the graph, as we zoom in, we can make $f(x)$ as close to 4 as we want, as long as x is sufficiently close to 2. We say, "The limit of $f(x) = x^2 - x + 2$ as x approaches 2 is 4." Mathematically,

$$\lim_{x \to 2}(x^2 - x + 2) = 4$$

Definition

Suppose $f(x)$ is defined when x is near a (but not necessarily at a). Then $\lim_{x \to a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to, but not equal to, a. We say "the limit of $f(x)$ as x approaches a is L."

Example 1. Use the following graphs to guess $\lim_{x \to a} f(x)$.



Example 2. Guess $\lim_{x\to 0} \dfrac{\sin x}{x}$.

Example 3. Guess $\lim_{x\to 0} \sin \dfrac{\pi}{x}$.

Example 4. The Heaviside function, named after Oliver Heaviside (1850 – 1925), is defined as

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

What is $\lim_{t\to 0} H(t)$?

Definition

We say that the **left-hand limit of** $f(x)$ **as** x **approaches** a is L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with $x < a$. Mathematically, $\lim_{x \to a^-} f(x) = L$.

Definition

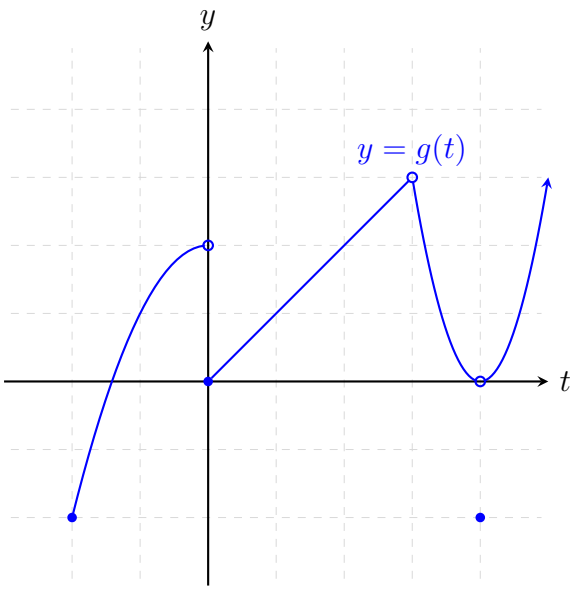
We say that the **right-hand limit of** $f(x)$ **as** x **approaches** a is L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with $x > a$. Mathematically, $\lim_{x \to a^+} f(x) = L$.

Example 5. Evaluate $\lim_{t \to 0^-} H(t)$ and $\lim_{t \to 0^+} H(t)$.

Definition

We say that $\lim_{x \to a} f(x) = L$ iff both $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Example 6. Given the graph of $y = g(t)$, find



a. $\lim_{t \to 0} g(t)$

b. $\lim_{t \to 0^-} g(t)$

c. $\lim_{t \to 0^+} g(t)$

d. $\lim_{t \to 3} g(t)$

e. $\lim_{t \to 4} g(t)$

Example 7. Use a table to find $\lim_{x\to 0} \frac{1}{x^2}$ if it exists.

Definition

Let f be a function defined on both sides of a. Then $\lim_{x\to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x to be sufficiently close to a.

Definition

Let f be a function defined on both sides of a. Then $\lim_{x\to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily negatively large by taking x to be sufficiently close to a.

Definition

The line $x = a$ is a **vertical asymptote** of $y = f(x)$ if at least one of the following statements is true:

- $\lim_{x\to a^-} f(x) = \infty$
- $\lim_{x\to a^-} f(x) = -\infty$
- $\lim_{x\to a^+} f(x) = \infty$
- $\lim_{x\to a^+} f(x) = -\infty$

Example 8. Write a limit that justifies that $\tan x$ has a vertical asymptote at $x = \frac{pi}{2}$.

4 Limit Laws

We will now learn to *evaluate* limits rather than *guess* them. The proofs of these laws are beyond the scope of this class, though many of them are accessible to think about.

Theorem: Limit Laws

Let $c \in \mathbb{R}$ and suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. The following are true:

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}((f(x)g(x)) = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$
5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, if $\sqrt[n]{a} \in \mathbb{R}$
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, if $\sqrt[n]{\lim_{x\to a} f(x)} \in \mathbb{R}$

Let's now use these limit laws to evaluate a limit rather than guess. Each step is justified by a particular limit law, so as we move through the next example, we will want to be able to look to a particular limit law to justify why we can move to the next step.

Example 1. Evaluate the limit. Justify each step.

(a) $\lim_{x\to 3}(2x^3 - 3x^2 + x + 2)$

(b) $\lim_{t\to 2}\left(\dfrac{t^2-2}{t^3-3t+5}\right)^2$

Even with all of these laws, we can still run into a wide range of difficulties.

Example 2. Evaluate the limit.

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

This last example demonstrates how the following proposition may be used.

Proposition

If $f(x) = g(x)$ whenever $x \neq a$, then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$ if the limit exists.

Example 3. Evaluate $\lim\limits_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

Example 4. Evaluate $\lim\limits_{t \to 2} \dfrac{\sqrt{4t+1} - 3}{t-2}$.

It turns out, each limit law applies to one-sided limits, as well.

Example 5. Prove $\lim_{x\to 4} \dfrac{|x-4|}{x-4}$ DNE.

Even with all of the strategies we've explored, many limits are still out of our reach. The Squeeze Theorem will be a strategy of finding a limit *indirectly* by bounding the expression we are trying to find the limit of.

The Squeeze Theorem

Suppose f, g, h are functions defined near a, except possibly at a. Suppose further that

1. $g(x) \leq f(x) \leq h(x)$ near a (except possibly at a)
2. $\lim_{x\to a} g(x) = L$, and
3. $\lim_{x\to a} h(x) = L$.

Then $\lim_{x\to a} f(x) = L$.

Example 6. Show $\lim_{x\to 0} x^2 \sin x = 0$.

5 Continuity

We've now seen a bunch of strategies for evaluating limits, but it does seem to be a very lengthy process. There must be a much more efficient method of evaluating limits, and that happens to be the case when a function is continuous.

We will now explore what makes a function continuous, what makes a function not continuous, special cases of continuity, and finally what functions are continuous.

Definition

A function f is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$.

That is, when a function is continuous at a, the limit is found by just evaluating the function at a.

Based on our definition, three things must be true for a function to be continuous:

(a) $f(a)$ must exist,

(b) $\lim_{x\to a} f(x)$ must exist, and

(c) $\lim_{x\to a} f(x) = f(a)$.

If any of these are not true, then we say that our function is discontinuous.

Definition

A function f defined near a is **discontinuous at** a when it is not continuous at a.

Example 1. Where are the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 + x - 20}{x + 5}$

(b) $F(x) = \begin{cases} \dfrac{x^2 + x - 20}{x + 5} & , x \neq -5 \\ 0 & , x = -5 \end{cases}$

(c) $g(x) = \begin{cases} \dfrac{1}{x^2} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

(d) $h(x) = \lfloor x \rfloor$

Continued...

The three kinds of discontinuities that arose in the previous example have different names.

(a) A **removable discontinuity** occurs when a function has a single point removed from what would otherwise be a continuous function.

(b) A **jump discontinuity** occurs when a function has a finite vertical gap between two otherwise continuous sides of a function.

(c) An infinite discontinuity occurs when a function has an infinite vertical gap (a vertical asymptote).

Example 2. Which kind of discontinuity did each of the functions in the previous example have?

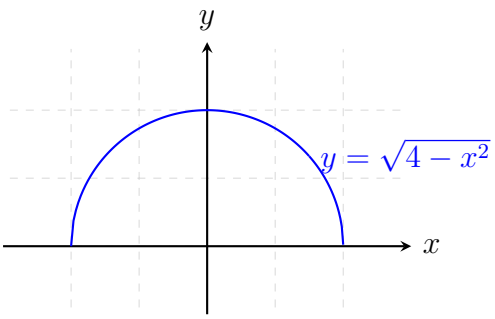
There are also other types of continuities. Just like limits, we can have one-sided continuity.

Definition

A function f is **continuous from the right at a number** a if $\lim_{x\to a^+} f(x) = f(a)$.
A function f is **continuous from the left at a number** a if $\lim_{x\to a^-} f(x) = f(a)$.
A function f is **continuous on an interval** if it is continuous at every number in the interval. Continuity at an endpoint is considered from a single side.

For example, at each integer, $\lfloor x \rfloor$ is continuous from the right but discontinuous from the left.

We will not be exploring one-sided continuity much. That said, we may say that $f(x) = \sqrt{4-x^2}$ is continuous on the interval $[-2, 2]$. What this means is that f is continuous on $(-2, 2)$, it is continuous from the right at -2, and it is continuous from the left at 2.



Now that we have an understanding of continuity on a large scale (not just at a point), we want to go back to our original purpose of continuity. That is, when a function is continuous at a, $\lim_{x\to a} f(x) = f(a)$, so evaluating limits is much easier.

So, what functions are continuous?

Theorem

If f and g are both continuous at a and $c \in \mathbb{R}$, then the following functions are continuous at a:

1. $f + g$
2. $f - g$
3. fg
4. $\frac{f}{g}$ if $g(a) \neq 0$
5. cf

Theorem

The following types of functions are continuous in their domains:

- Polynomials
- Rational Functions
- Root Functions
- Exponential Functions
- Logarithmic Functions
- Trigonometric Functions
- Inverse Trigonometric Functions

Example 3. Find $\lim_{x\to -1} \dfrac{x^2+1}{2x^2-x-1}$.

Example 4. Where is $f(x) = \dfrac{\ln x + \arctan x}{x^2-1}$ continuous?

The previous theorems said that most every parent function you've encountered previously are continuous where they are defined. Moreover, adding, subtracting, multiplying, and subtracting continuous functions produces a continuous function. It turns out, composing continuous functions also produces a continuous function.

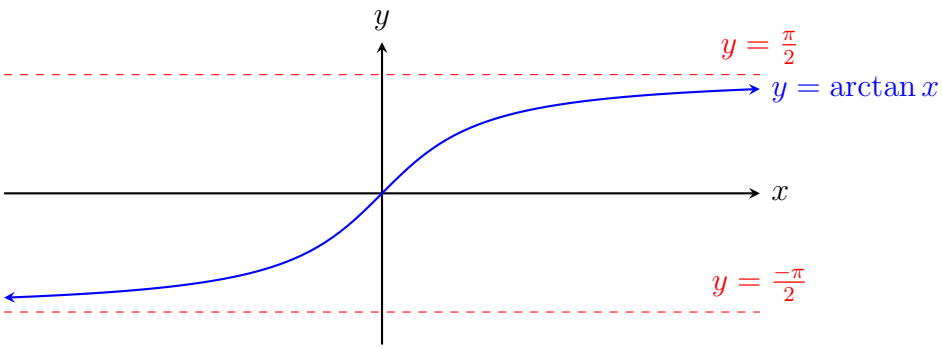
Theorem

If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Example 5. Evaluate $\lim_{x\to \pi} \sin(x + \sin x)$.

6 Limits Involving Infinity

Consider the graph of the function $f(x) = \arctan x$. Notice that the larger x gets, the closer $f(x)$ gets to $\frac{\pi}{2}$.



Definition

Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large.

Definition

Let f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x\to-\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large negatively.

Thus, we can write

$$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2} \qquad \text{and} \qquad \lim_{x\to\infty} \arctan x = \frac{\pi}{2}$$

We have seen this behavior before. The language of limits allows us to state this formal definition.

Definition

The line $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L$$

Question: Can the graph of a function cross a horizontal asymptote?

Question: Can the graph of a function cross a vertical asymptote?

Example 1. Find $\lim_{x\to-\infty}\frac{1}{x}$ and $\lim_{x\to\infty}\frac{1}{x}$.

Note that all Limit Laws (except 9: $\lim_{x\to a}x^n=a^n$ and 10: $\lim_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$) hold for limits at ∞. Combining the Limit Laws with the previous Example, we get the following theorem.

Theorem

If $r>0$ is a rational number, then $\lim_{x\to\infty}\frac{1}{x^r}=0$. Moreover, if x^r is defined for all x, then $\lim_{x\to-\infty}\frac{1}{x^r}=0$.

This theorem will be the primary strategy for evaluating limits at infinity.

Example 2. Evaluate $\lim_{x\to\infty}\frac{4x^3+6x^2-2}{2x^3-4x+5}$.

Example 3. Find all vertical and horizontal asymptotes of $\lim_{t\to-\infty} \dfrac{\sqrt{9t^6 - t}}{t^3 + 1}$.

Example 4. Evaluate $\lim_{s\to\infty} \sqrt{4s^2 + 1} - 2s$.

Recall that the graph of e^x has an asymptote. Where?

Proposition

$$\lim_{x\to-\infty} e^x = 0.$$

Example 5. Evaluate $\lim_{x\to0^-} e^{\frac{1}{x}}$.

Example 6. Evaluate $\lim_{\theta\to\infty} \cos\theta$.

7 Derivatives

Previously, we saw that we could estimate the slope of a tangent line and the instantanenous rate of change by using average rates of change. Our goal is to be able to *calculate* the slope of a tangent line and *calculate* an instantaneous rate of change using limits.

Recall the **average rate of change** of a function f between $x = a$ and $x = b$ is given by $\dfrac{f(b) - f(a)}{b - a}$. If we fix a and let b vary, then we can allow b to get closer and closer to a. From §2.1 we saw that the closer b got to a, the more accurate estimate of the slope of a tangent line got.

Definition

The **tangent line** to the curve $y = f(x)$ at $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

given that the limit exists.

Example 1. Find an equation of the tangent line to the curve $y = x^2$ at $P(1, 1)$.

Lines are characterized by having a slope, but if we refer to the **slope of the curve at the point** P, we mean the "slope of the tangent line to the curve at the point P."

Example 2. Find the slope of the curve $y = x^3 - 3x + 1$ at $(2, 3)$.

Now, instead of looking at average rate of change being the average of two different points, let's change our perspective. Consider average rate of change as being one point $(a, f(a))$ and another a bit more than a, which we will call $(a+h, f(a+h))$. Then the **slope of the tangent line to** $f(x)$ **at** $(a, f(a))$ is

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{a+h-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Definition

The **tangent line** to the curve $y = f(x)$ at $P(a, f(a))$ is the line through P with slope

$$\begin{aligned} m &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \end{aligned}$$

given that the limit exists.

Example 3. Find the slope of the tangent line to $f(x) = \dfrac{3}{x+1}$ at $(2, 1)$.

Previously, we discussed the idea of dropping a ball off of the CN Tower and investigated its instantenous velocity at 5 seconds. We found that the average velocity was given by $\frac{f(a+h)-f(a)}{h}$, in the interval $[a, a+h]$ and we made the interval (size h) smaller and smaller.

Definition

If $s(t)$ gives the displacement of an object at time t, then the **velocity** $v(a)$ of that object at time $t = a$ is the limit of average velocities as the interval lengths approach zero:

$$v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Example 4. A ball is dropped from the CN Tower 450 m above ground. The displacement the ball travels is given by $s(t) = 4.9t^2$. What is the velocity of the ball after 5 seconds?

Example 5. How fast is the ball going when it hits the ground?

Because the same kind of calculation appears in each of these exercises, so we unite them with a single word.

Definition

The **derivative of a function f at a number** a, denoted $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 6. Find the derivative of $f(x) = x^2 - 7x + 1$ at a. At 3.

Example 7. Find the derivative of $g(x) = \sqrt{x - 1}$ at 10.

Example 8. Find an equation of the tangent line to the curve $y = x^2 - 7x + 1$ at $(3, -11)$.

Example 9. The number of bacteria t hours in a controlled laboratory experiment is $n = f(t)$. What does $f'(5)$ mean? What are its units? Which is bigger: $f'(5)$ or $f'(10)$?

8 The Derivative Function

We just learned that the derivative of f at a is $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$.

Instead of taking the derivative at a fixed number, we can consider taking the derivative in varying places. Effectively, we are replacing the constant a with the variable x.

Definition

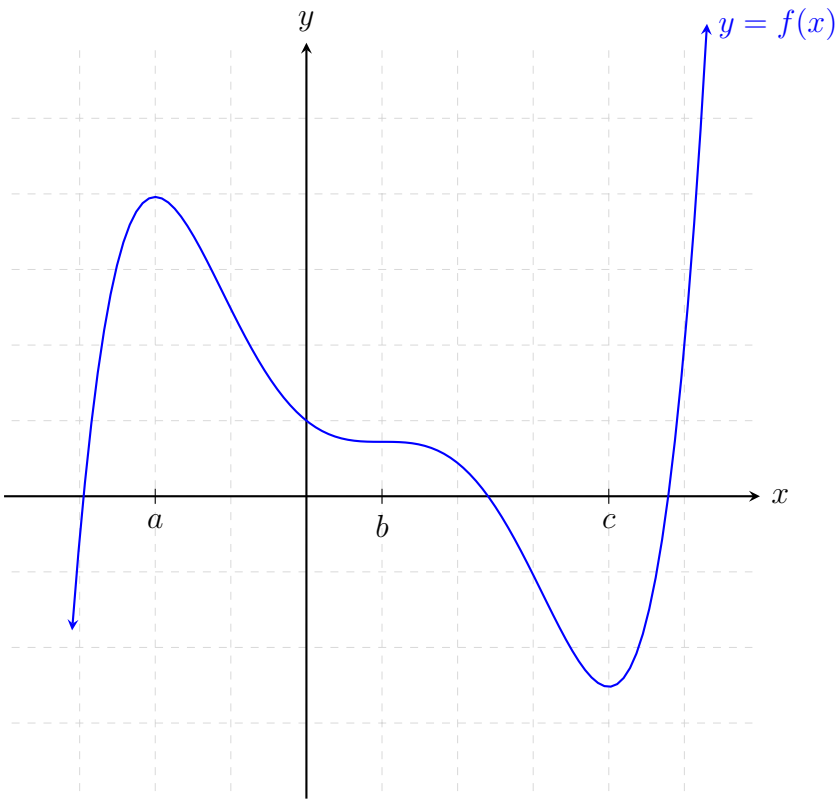
For any number x such that the limit exists, the **derivative of** f is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The function f' is called the *derivative* of f because it has been "derived" from f by taking a particular limit. Moreover, because we have a new function, we consider its domain. The domain of f' is $\{x \mid f'(x) \text{ exists}\}$.

Now that we have the domain, let's learn how to sketch the graph of a derivative.

Example 1. Sketch f' given the graph of $y = f(x)$.



Example 2. If $f(x) = x^2 - 7x + 1$, find a formula for $f'(x)$.

Example 3. If $g(x) = \sqrt{x}$, find a formula for $g'(x)$. What is the domain of g'?

Newton may be the Father of Calculus, but Gottfried Wilhelm Leibniz (1646 – 1716) independently invented calculus and even published before Newton. Leibniz spent a good part of his life studying and developing symbolic logic and notation in order to simplify mathematics. In particular to our studies, Leibniz invented *differential operators*.

Definition

The process of calculating a derivative is called **differentiation** and uses notation called **differential operators**. If $y = f(x)$, then the following are equivalent notations:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The operators D and $\frac{d}{dx}$ are called differential operators because they are instructions that tell us to differentiate. We call y' and f' **prime notation**, and we call $\frac{d}{dx}$ and $\frac{dy}{dx}$ **Leibniz notation**.

One advantage that prime notation has over Leibniz notation is evaluation. In Leibniz notation, $\left.\frac{dy}{dx}\right|_{x=a} = f'(a)$.

Example 4. Find $\frac{dy}{dx}$ if $y = \dfrac{3}{x+1}$.

Definition

A function f is **differentiable at** a if $f'(a)$ exists. It is differentiable on an interval if it is differentiable at every number in the interval.

Example 5. Where is the function $f(x) = |x|$ differentiable?

Theorem

If f is differentiable at a, then f is continuous at a.

Please note that the converse is *not* true.

Example 6. Sketch the graph of a function that is continuous at a but not differentiable at a.

We can restate the previous theorem to help us to identify what it means for a function to not be differentiable.

Theorem

If f is not continuous at a, then f is not differentiable at a.

There are a few reasons that a function may not be differentiable. Thus, a function f is not differentiable at a when any of the following apply.

- f is not continuous at a.
- The graph of f has a corner or cusp (abrupt change in direction $\Rightarrow$ No tangent).
- Vertical tangent line.

Definition

A curve has a **vertical tangent line** when $x = a$ if f is continuous at a and $\lim_{x \to a} |f'(x)| = \infty$.

Definition

If f is a differentiable function, then the derivative of f' is called the **second derivative** of f and is denoted f''. Using Leibniz, the second derivative of $y = f(x)$ is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 7. Find the second derivative of $y = x^2 - 7x + 1$.

The most common application we will use of multiple derivatives is in regards to an object in motion. If s is the position of an object, v is velocity, and a is acceleration, then.

$$\begin{aligned} \text{Position:} \quad & s(t) \\ \text{Velocity:} \quad & v(t) = \frac{d}{dt}(s(t)) \\ \text{Acceleration:} \quad & a(t) = \frac{d}{dt}(v(t)) = \frac{d^2}{dt^2}(s(t)) \end{aligned}$$

9 Derivatives of Polynomial & Exponential Functions

Now that we've learned all about derivatives, we may notice that we are limited by how long it takes to evaluate a derivative. There are also lots of derivatives we may not be able to compute due to limitations around algebra. We will spend a good amount of time shifting our focus to finding a more efficient method of evaluating derivatives.

Derivative of a Constant

$$\frac{d}{dx}(c) =$$

Consider $f(x) = c$. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$=$$

$$=$$

$$=$$

$$=$$

Derivative of x

$$\frac{d}{dx}(x) =$$

Consider $f(x) = x$. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$=$$

$$=$$

$$=$$

$$=$$

Derivative of x^2

$\frac{d}{dx}(x^2) =$

Consider $f(x) = x^2$. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

We notice that

$$\frac{d}{dx}(x^0) = 0$$
$$\frac{d}{dx}(x^1) = 1x^0$$
$$\frac{d}{dx}(x^2) = 2x^1$$
$$\frac{d}{dx}(x^3) =$$
$$\frac{d}{dx}(x^4) =$$
$$\vdots$$
$$\frac{d}{dx}(x^n) =$$

The Power Rule

If n is a positive integer, then $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

We differentiated $\sqrt{x}$ using the limit definition of derivative. It turned out that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. In exponential form, $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$. We may notice that the Power Rule works for any real number power n.

The General Power Rule

If $n \in \mathbb{R}$, then $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

Example 1. Differentiate the following functions.

(a) $f(x) = x^6$

(b) $g(x) = \frac{1}{\sqrt[3]{x}}$

(c) $h(x) = \frac{1}{x^4}$

The Constant Multiple Rule

If $c \in \mathbb{R}$ and f is a differentiable function, then $\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$.

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h\to 0} c\, \frac{f(x+h) - f(x)}{h} \\ &= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= cf'(x) \end{aligned}$$

□

Example 2. Evaluate $\frac{d}{dx}(6x^6)$

The Sum Rule

If f and g are differentiable, then $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.

Proof. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h\to 0} \frac{(f(x+h) - f(x) + g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

□

By a parallel proof, we can establish the following rule.

Difference Rule

If f and g are differentiable, then $\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$.

By combining the Constant Multiple, Sum, Difference, and Power Rules, we can differentiate any polynomial.

Example 3. Differentiate $7x^5 - 2x^4 + 12x^3 + 1.9x^2 - \pi x + e$.

Example 4. A particle is moving back and forth on a line. The equation of motion for the particle is given by $s = t^4 - 2t^3 + t^2 - t$, where t is seconds and s is meters. Find the velocity and acceleration as a function of t, and find the acceleration of the particle after 1 s.

Feel free to use this Desmos applet to visualize this motion.

Let's explore the derivative of an exponential function. Consider $f(x) = a^x$, where $a > 0$, $a \neq 1$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \end{aligned}$$

Notice that $f'(0) = a^0 \lim_{h\to 0} \frac{a^h - 1}{h}$, so $f'(0) = \lim_{h\to 0} \frac{a^h - 1}{h}$.

We don't have an explicit way to compute this, so let's go back to tabular form to try to guess this limit. That is, try different values of h getting closer to 0 while also trying different a-values. What we will look for is the specific value of a so that this limit gives 1. In that particular case, we will have

$$\frac{d}{dx}(a^x) = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x$$

Explore these values in this Desmos applet.

Definition

The number e is the unique number such that $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

Example 5. Differentiate $g(r) = e^r - r^e$.

10 The Product & Quotient Rules

We have seen that $\frac{d}{dx}(f(x) \pm g(x))' = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$. These same rules do not apply to products and quotients. Unfortunately, these derivative formulas are a bit more involved.

The Product Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$

or rather $(fg)' = fg' + gf'$.

Proof. Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{g(x)(f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

□

Example 1. If $f(x) = xe^x$, find $f'(x)$. Then find $f^{(n)}(x)$.

The Quotient Rule

If f and g are differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$

or rather $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$.

Proof. Let $F(x) = \frac{f(x)}{g(x)}$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \frac{f(x+h)g(x) - f(x)g(x+h)}{h} \\
&= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \\
&= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \\
&= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \left(\lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} + \lim_{h\to 0} \frac{f(x)g(x) - f(x)g(x+h)}{h}\right) \\
&= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \left(g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) \\
&= \frac{1}{(g(x))^2}(g(x)f'(x) - f(x)g'(x)) \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}$$

□

Example 2. Let $y = \frac{x+1}{x^3+x-2}$. Find $\frac{dy}{dx}$.

Example 3. Find an equation of the tangent line to the curve $\frac{e^x}{1+x^2}$ at the point $(1, \frac{1}{2}e)$.

Example 4. A particle moves on a line for 1 second, and its velocity is given in feet per second by $v(t) = \sqrt{t}(2 - 3t + t^2)$. At what time is the particle not accelerating?

11 Derivatives of Trigonometric Functions

In order to differentiate the trigonometric functions, we need to evaluate a couple limits. One of them will be offered without proof. The other has the proof attached.

Lemma

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$
$$\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$$

Proof. We will prove that $\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$.

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\cdot\frac{\cos\theta+1}{\cos\theta+1}\right)\\
&= \lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)}\\
&= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}\\
&= -\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta+1}\right)\\
&= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta+1}\\
&= -1\cdot\left(\frac{0}{1+1}\right)\\
&= 0
\end{aligned}$$

□

We will use these limits along with the addition formula for sine to find a formula for the derivative of $\sin x$. Recall that

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$$

Derivative of Sine

$$\frac{d}{dx}(\sin x) =$$

Proof. Let $F(x) = \sin x$. Then

$$\frac{d}{dx} f(x)x =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

□

Example 1. Differentiate $y = e^x \sin x$.

Using a parallel argument to differentiating Sine, we can find the derivative of cosine.

Derivative of Cosine

$$\frac{d}{dx}(\cos x) = -\sin x$$

Example 2. Find a formula for $\frac{d}{dx}(\tan x)$.

Derivative of Tangent

$$\frac{d}{dx}(\tan x) =$$

Using the Quotient Rule, we can differentiate all of the trigonometric functions.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example 3. If $f(x) = \dfrac{1 - \sec x}{\tan x}$, find $f'(x)$.

Example 4. If $f(t) = \csc t$, find $f''\left(\frac{\pi}{6}\right)$.

Example 5. Find the 27th derivative of $\cos x$.

Example 6. A weight at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. Its position at time t is $s(t) = 4\cos t$.

(a) Find the velocity of the weight at time t.

(b) Find the acceleration of the weight at time t.

(c) When is the weight moving the fastest?

(d) When is the weight not moving?

(e) When is the weight accelerating fastest?

12 The Chain Rule

We can differentiate f, cf, $f+g$, $f-g$, fg, and $\frac{f}{g}$, provided we know the derivative of f and g. Now, we learn to differentiate $f \circ g$. We will present this without proof.

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $F(x) = (f \circ g)(x) = f(g(x))$ is differentiable at x and

$$F'(x) = f'(g(x)) \cdot g'(x)$$

Using Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Leibniz is nice because it shows that the derivative of a composition is the product of the derivatives. That said, it is difficult to use Leibniz notation to do inline computations.

Example 1. Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Example 2. Differentiate $y = (3x^2 - 4)^{29}$.

Example 3. Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[5]{sinx+x^2}}$.

Example 4. Differentiate $f(x) = e^{\sin x}$.

Example 5. Differentiate $y = \sin(\cos(\tan x))$.

Let's try differentiating $y = a^x$ again. We know how to differentiate e^x, and we can rewrite $a = e^{\ln a}$. Thus,

$$\begin{aligned}\frac{d}{dx}a^x &= \frac{d}{dx}\left(e^{\ln a}\right)^x \\ &= \frac{d}{dx}\left(e^{(\ln a)x}\right) \\ &= e^{(\ln a)x} \cdot \frac{d}{dx}(\ln a)x \\ &= e^{(\ln a)x} \cdot \ln a \\ &= a^x \ln a\end{aligned}$$

Derivative of an Exponential Function

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Example 6. Differentiate $y = 3^x$.

Example 7. Find $f'(e^{50})$ if $f(x) = 10^x$.

13 Implicit Differentiation

Up to now, we've been differentiating functions. Other graphs can also certainly have tangent lines, too – such as circles, ellipses, hyperbolas, and other non-functions.

Definition

An equation that is written in the form of one variable is equal to an expression of the other(s) is an **explicit equation**. An equation that has mixed-variable expressions is an **implicit equation**.

For example, $y = 2x^2 + \sqrt{3 - x^3}$ is an example of an explicit equation, while $4x^2 - 25y^2 = 100$ is written implicitly.

When we differentiate $y = f(x)$, we are noting that y is a function of x, and we can write the following.

$$\begin{aligned} y &= f(x) \\ \frac{d}{dx}(y) &= \frac{d}{dx}(f(x)) \\ \frac{dy}{dx} &= \frac{d}{dx}(f(x)) \end{aligned}$$

When we differentiate an implicit function with respect to x, we assume that each other variable is a function of x. Thus, when we differentiate an implicit equation, we must use the *chain rule*.

Consider these side-by-side examples.

$$\frac{d}{dx}(\sin(x^2+1)) = \cos(x^2+1)\ \frac{d}{dx}(x^2+1) \qquad \frac{d}{dx}(\sin(y)) = \cos(y)\frac{d}{dx}(y)$$

The second equation comes from rewriting the first equation with $y = x^2 + 1$. This demonstration is showing us that when we differentiate an expression with respect to a different variable, we must use the chain rule.

Example 1. If $x^2 + y^2 = 4$, find $\frac{dy}{dx}$.

Example 2. Find an equation for the line tangent to $x^2 + y^2 = 4$ at $(\sqrt{3}, -1)$.

Example 3. Find y'' if $x^4 + y^4 = 16$.

Example 4. Find y' if $x^3 + y^3 = 6xy$. At what x-value in the first quadrant is the tangent line horizontal?

14 Derivatives of Inverse Trigonometric Functions

It's been a bit since we've learned about the inverse trigonometric functions, so let's start by reviewing some definitions of two of them.

We first need to acknowledge that $y = \sin x$ is not an invertible function. In order for $\sin x$ to be invertible, it must pass the horizontal line test. So fix this, we restrict the domain of sine to create the restricted sine function.

$$y = \sin x \qquad , \qquad \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$$

This function is invertible because it passes the horizontal line test. To find that inverse, we switch the x and the y to produce the following.

$$x = \sin y \qquad , \qquad \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$$

This defines the inverse for sine.

Definition

We define the **arcsine** function as $y = \arcsin x$ iff $\sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Now that we understand what the arcsine function is, let's find a derivative for it.

Derivative of Arcsine

$$\frac{d}{dx}(\arcsin x) =$$

We can do a very similar workflow for the arctangent function.

Definition

We define the **arctangent** function as $y = \arctan x$ iff $x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

If we follow the same process, we will arrive at the derivative below.

Derivative of Arctangent

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Moreover,

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$

The inverses of Sine and Tangent are the most common, but the others *may* pop up from time to time.

Example 1. Find $\frac{d}{dx}\left(\sin^{-1} 3x\right)$.

Example 2. Find y' if $y = \cos^{-1}(\sin 2\theta)$.

Example 3. Differentiate $f(x) = \arctan\sqrt{\dfrac{1-x}{1+x}}$.

15 Derivatives & Logarithms

In the previous section, we saw how to find a derivative formula for inverse trigonometric functions. The way that we did that was to differentiate a known function with the x and y swapped. Let's now apply that same strategy to find a derivative of a logarithmic function.

We know that if $y = a^x$, then $\frac{dy}{dx} = a^x \ln a$. If we want to find an inverse of the exponential function a^x, then we swap x and y to get $x = a^y$.

Definition

We define the **logarithm with base** a as $y = \log_a x$ iff $a^y = x$, where $a > 0$ and $a \neq 1$.

Derivative of a Logarithm

$$\frac{d}{dx}(\log_a x) =$$

Now that we have a formula for $\frac{d}{dx}(\log_a x)$, we can say that

$$\begin{aligned}\frac{d}{dx}(\ln x) &= \frac{d}{dx}(\log_e x) \\ &= \frac{1}{x \ln e} \\ &= \frac{1}{x}\end{aligned}$$

Derivative of the Natural Logarithm

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

By far the more common function to see is the natural logarithmic function, but logarithms with other bases do occur from time to time.

Example 1. Differentiate $y = \ln(\arctan x)$.

Example 2. Find $f'(x)$ if $f(x) = \sqrt[3]{\ln x}$.

Example 3. Find $\frac{dy}{dx}$ if $y = \log(xe^x)$.

Example 4. Find $f'(x)$ if $f(x) = \ln|x|$.

This derivative doesn't come up for awhile, but it is paramount to a development in Integral Calculus. That is,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

You may have seen before that e is defined as the limiting value of $\left(1+\frac{1}{n}\right)^n$ as $n \to \infty$. But why is this? Usually, it is presented with tables. With what we know now, let's prove that $e = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n$.

Theorem

The number $e = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n$.

Proof. If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$. By the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{x\to 0} \frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0} \frac{\ln(1+x)-\ln 1}{x} \\ &= \lim_{x\to 0} \left[\frac{1}{x}\left(\ln(1+x)\right)\right] \\ &= \lim_{x\to 0} \ln\left(1+x\right)^{\frac{1}{x}} \end{aligned}$$

Since $f'(1) = 1$, $\lim\limits_{x\to 0} \ln(1+x)^{\frac{1}{x}} = 1$. It follows that

$$\begin{aligned} e &= e^1 \\ &= e^{\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}}} \\ &= \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} \\ &= \lim_{x\to 0} (1+x)^{\frac{1}{x}} \end{aligned}$$

Moreover, if $n = \frac{1}{x}$, then $n \to \infty$ as $x \to 0^+$, so by substitution,

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

□

16 Related Rates

Calculus is about the study of change and how one change is related to another change. Previously, we have studied changes of y with respect to x, how a function changes with respect to its independent variable, and how changes in displacement affect velocity and acceleration.

In this section, we study how changes in one quantity affects changes in another quantity. That is, we express rates of change of one thing in terms of rates of change of another.

The procedure of related rates is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

In order to work through related rates prompts, we will follow this process.

(a) Draw a picture from the prompt. Include variables.

(b) Write down Know and Want.

(c) Reading the prompt, write what we know and want in differential operator notation such as $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

(d) Write an equation relating the quantities, such as between x and y. This often comes from geometry.

(e) Differentiate the equation with respect to time.

(f) Solve for the quantity you Want.

(g) Evaluate the differentiated equation with the initial conditions.

(h) Write your conclusion in the form of a sentence with units.

Example 1. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Example 2. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall.

Example 3. An ice cream cone has the shape of an inverted circular cone with base radius 2 inches and height 4 inches. If ice cream is being pumped into the cone at a rate of 2 cubic inches per second, find the rate at which the ice cream is rising when the ice cream is 3 inches deep.

Example 4. A car is traveling west at 50 mph, and a motorcycle is traveling north at 60mph. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when the car is 0.3 miles and the motorcycle is 0.4 miles from the intersection?

17 Extrema

Possibly the most important application of differential calculus is optimization. Many of these prompts can be reduced to finding the maximum or minimum value of a function.

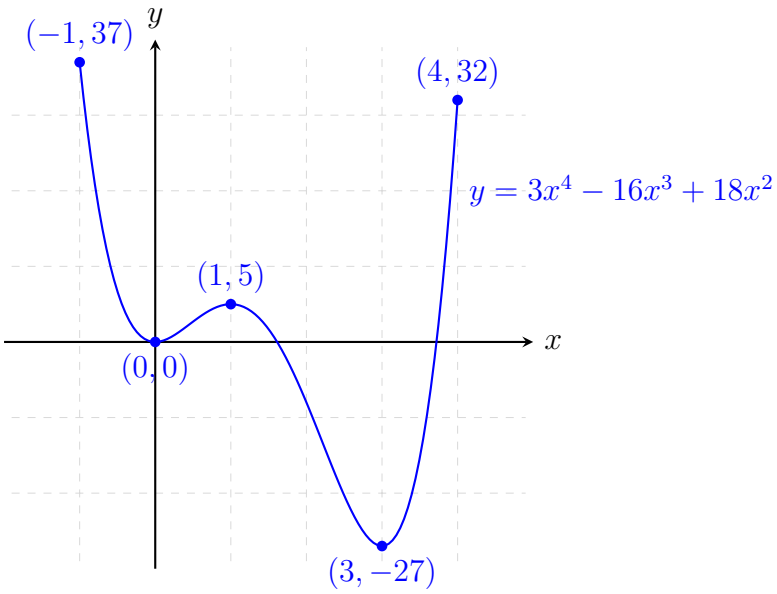
Definition

If f is a function with domain D and $c \in D$, then $f(c)$ is:

- A **local maximum** of f if $f(c) \geq f(x)$ for all x near c.
- A **local minimum** of f if $f(c) \leq f(x)$ for all x near c.
- The **absolute (global) maximum** of f if $f(c) \geq f(x)$ for all $x \in D$.
- The **absolute (global) minimum** of f if $f(c) \leq f(x)$ for all $x \in D$.

The plural of maximum is **maxima**. The plural of minimum is **minimum**. An **extremum** is either a maximum or a minimum. The plural of extremum is **extrema**.

Example 1. What are the local extrema and global extrema of $f(x) = 3x^4 - 16x^3 + 18x^2$?



Example 2. What are the extrema of the absolute value function?

Example 3. What are the extrema of the cubic function?

Example 4. What are the extrema of the cubic function restricted to $[-1, 2]$?

As this last example shows, some functions have extrema while others do not. The Extreme Value Theorem tells what functions have extrema.

Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a.b]$.

Notice that this is not an "if and only if" theorem. Other functions have extrema that are not continuous.

In previous courses, the discussion of extrema would end right here, but we now have derivatives. Something we may notes is that the slope of a continuous function at an extremum is either 0 or undefined. This observations is the statement of a theorem attributed to one of the most famous mathematicians of the 17th century, Pierre de Fermat (1601 – 1655). Fermat was a French lawyer that predated Newton and calculus. He is perhaps most well-known contemporarily for Fermat's Last Theorem, a conjecture written in the margin of Diophantes' *Arithmetica* that took nearly 500 years to prove.

Fermat's Theorem

If f has a local extrema at c and $f'(c)$ exists, then $f'(c) = 0$.

Fermat's Theorem is very powerful, but just like the Extreme Value Theorm, it is not an "if and only if". Just because $f'(c) = 0$ doesn't guarantee that $f(c)$ is a local extremum. Moreover, not all extrema have $f'(c) = 0$.

In general, we cannot locate extrema simply by solving $f'(x) = 0$. However, this is a great place to begin looking.

Definition

A **critical number** of a function f is a number c in the domain of f such that either

- $f'(c) = 0$ *or*
- $f'(c)$ does not exist

Example 5. Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4 - x)$.

Fermat's Theorem can be rewritten with this new terminology.

Fermat's Theorem

If f has a local extremum at c, then c is a critical number of f.

This provides a strategy to look for absolute extrema (*not* local) on a closed interval.

Closed Interval Method for Finding Global Extrema

To find the global extrema of a continuous function f on the closed interval $[a, b]$,

1. Find the critical numbers in (a, b)
2. Find $f(c)$ for all of the critical values c
3. Find $f(a)$ and $f(b)$
4. The largest value of the absolute max; the least value is the absolute min

Example 6. Find the absolute max and min of $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \leq x \leq 4$.

18 Mean Value Theorem

The Mean Value Theorem

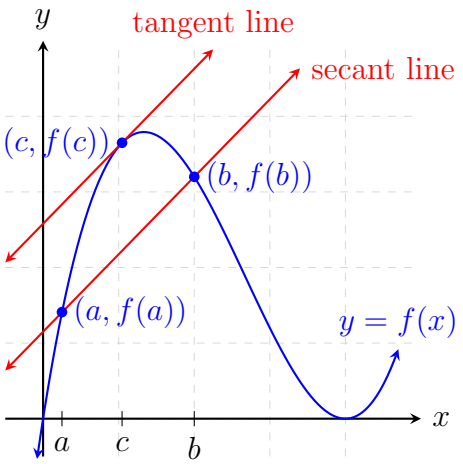
Let f be a function with the following properties.

(a) f is continuous on $[a, b]$

(b) f is differentiable on (a, b)

Then there exists some number $c \in (a, b)$ such that either

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{or} \qquad f(b) - f(a) = f'(c)(b - a)$$



The Mean Value Theorem allows us to extract information about graphs from functions.

As we've seen in the past couple sections, the derivative of a function tells us a lot about a function. Because f' represents the slope of the curve $y = f(x)$, it tells us where f will proceed at each point.

Increasing/Decreasing Test

If $f'(x) > 0$ on an interval, then f is increasing on that interval. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof. Suppose $f'(x) > 0$ on an interval. Let x_1, x_2 be any two numbers in the interval with $x_1 < x_2$. We want to show that $f(x_1) < f(x_2)$. Since $f'(x) > 0$, we know that f is differentiable on $[x_1, x_2]$. By the Mean Value Theorem, there is a number $c \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. By assumption, $f'(c) > 0$, and $x_2 - x_1 > 0$ since $x_1 < x_2$. Thus, $f'(c)(x_2 - x_1) > 0$, so $f(x_2) - f(x_1) > 0$. Hence, $f(x_1) < f(x_2)$, and f is increasing.

Decreasing is parallel argument. □

Example 1. Find the intervals of increase and decrease for $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Recall that Fermat's Theorem states that if f has a local extremum at c, then c must be a critical number of f. We have also seen that not every critical number gives rise to a local extremum. The following theorem states when a critical number corresponds to an extremum or not.

First Derivative Test

Suppose that c is a critical number of a continuous function f.

- If f' changes from positive to negative at c, then f has a local maximum at c
- If f' changes from negative to positive at c, then f has a local minimum at c
- If f' does not change sign at c, then f has no local extremum at c

This test is a direct consequence of the Increasing/Decreasing Test and thus requires no proof.

Example 2. Find the local extrema of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Example 3. Find the local extrema of $g(x) = x^2 \ln x$.

Example 4. Find the local extrema of $h(x) = x + 2\sin x, 0 \leq x \leq 2\pi$.

We have done a lot of exploration of how the derivative of a function and the original function interact. The first derivative determines when a function f increases, decreases, levels off, and has an extremum. It turns out, the second derivative determines when a function f is concave up, is concave down, and has a point of inflection.

Definition

If the graph of f lies above all of its tangents on an interval I, then f is called **concave upward** on I. If the graph of f lies below all of its tangents on I, then f is called **concave downward** on I.



Notice that in the graph on the left, the slope of a tangent line to the graph of $y = f(x)$ increases as x increases. See this happen here. Similarly, in the second graph, the slope of the tangent line to the graph of $y = f(x)$ decreases as x increases. This can be seen here.

In the first graph, this means that $\dfrac{dy}{dx}$ increases as x increases. By the Increasing/Decreasing Test, this means that $\dfrac{d^2y}{dx^2}$ is positive as x increases. Since this graph is a concave up graph, this leads us to believe that when $f''(x) > 0$, f is concave up. Similarly, when $f''(x) < 0$, f is concave down.

Concavity Test

If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I. If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

Definition

A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous and changes concavity at that point.

Example 5. Sketch the graph of a function f that satisfies the following properties:

- $f'(x) > 0$ on $(-\infty, 1)$,
- $f'(x) < 0$ on $(1, \infty)$,
- $f''(x) > 0$ on $(-\infty, -2), (2\infty)$,
- $f''(x) < 0$ on $(-2, 2)$
- $\lim_{x\to-\infty} f(x) = -2$, and
- $\lim_{x\to\infty} f(x) = 0$

Another consequence of the First Derivative Test and the Concavity Test is that extrema can be found using the second derivative.

Second Derivative Test

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c

Example 6. Find the intervals on which f is increasing or decreasing, the local extrema of f, the intervals of concavity, and the inflection points where $f(x) = x^4 - 4x^3$.

19 Optimization

Optimizing a Function

1. Understand the situation, and draw a picture.
2. Define the variables.
3. Decide what we want to **optimize**, and write an equation that models the quantity.
4. If there is a **constraint**, write an equation that models the constraint.
5. Substitute the **constraint** into the **optimizing equation** to create a single-variable function that represents the quantity we want to optimize.
6. Determine if there is a domain restriction for which the optimizing function is valid.
7. Use Calculus (the Closed Interval Method, First-Derivative Test, and/or Second-Derivative Test) to optimize the function. Make sure to verify that your conclusion is inside the valid domain.
8. State your conclusion.

Example 1. A company wishes to sell 1 L cans of olive oil. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can. *Hint: 1 L = 1000 cm^3.*

Example 2. A rectangle is inscribed inside of a circle with radius r. Prove that the rectangle with the greatest area is a square.

20 L'Hôpital's Rule

20.1 Indeterminate Forms

We have learned several strategies for evaluating a limit. Use what you have previously learned to evaluate the following limit.

Example 1. Evaluate $\lim_{x\to 1} \dfrac{x^2-1}{x^2+x-2}$

The limit $\lim_{x\to 1} \dfrac{\ln x}{x-1}$ exists. We want to use the quotient law, but we can't, as the denominator's limit is 0. We cannot factor or use any of the other strategies we've learned before. Knowing the limit exists, how do we compute it?

Definition

In general, if we have $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$, then this limit may or may not exist and is called an **indeterminate form of type** $\frac{0}{0}$.

Definition

In general, if we have $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$, then this limit may or may not exist and is called an **indeterminate form of type** $\frac{\infty}{\infty}$.

20.2 L'Hôspital's Rule

L'Hôspital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a). Suppose that either

- $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, or
- $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

(that is, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit exists on the right side (or is $\pm\infty$).

Note that

1. We *must* have an indeterminate form in order to use L'Hôspital's Rule
2. L'Hôspital's Rule is also valid for one-sided limits and limits at infinity

Example 2. Find $\lim_{x\to 1} \dfrac{\ln x}{x-1}$.

Example 3. Find $\lim_{x\to\infty} \dfrac{\ln x}{x-1}$.

Example 4. Find $\lim_{t\to\infty} \dfrac{e^t}{t^2}$.

Example 5. Find $\lim_{x\to 0} \dfrac{\tan x - x}{x^3}$.

Example 6. Find $\lim_{x\to \pi^-} \dfrac{\sin x}{1-\cos x}$.

20.3 Another Indeterminate Form

Definition

In general, if we have $\lim_{x\to a} f(x)g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$, then this limit may or may not exist and is called an **indeterminate form of type** $0 \cdot \infty$.

Example 7. Evalute $\lim_{x\to 0^+} x \ln x$.

Example 8. Evaluate $\lim_{t\to\infty} e^{-t}t^2$.

There are other indeterminate forms, as well, but we will not focus on those at this time.