

MTH 252Z

Midterm Review Key

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1. Find the most general antiderivative of $f(x) = 8x^9 - 3x^6 + 12x^3$.

Solution:

$$\begin{aligned} F(x) &= \frac{8}{10}x^{10} - \frac{3}{7}x^7 + \frac{12}{4}x^4 + C \\ &= \frac{4}{5}x^{10} - \frac{3}{7}x^7 + 3x^4 + C \end{aligned}$$

2. Find the most general antiderivative of $f(t) = \sin t + 2 \cos t$.

Solution: $F(t) = -\cos t - 2 \sin t + C$

3. Find f if $f'(t) = 5t^4 - 3t^2 + 4$ and $f(-1) = 2$.

Solution:

$$\begin{aligned} f(t) &= \frac{5}{5}t^5 - \frac{3}{3}t^3 + 4t + C \\ &= t^5 - t^3 + 4t + C \\ 2 &= f(-1) \\ &= (-1)^5 - (-1)^3 + 4(-1) + C \\ &= -4 + C \implies C = 6 \\ f(t) &= t^5 - t^3 + 4t + 6 \end{aligned}$$

4. Find f if $f''(x) = 8x^3 + 5$ and $f(1) = 0$, $f'(1) = 8$.

Solution:

$$\begin{aligned} f'(x) &= \frac{8}{4}x^4 + 5x + C \\ &= 2x^4 + 5x + C \\ 8 &= f'(1) \\ &= 2(1)^4 + 5(1) + C \\ &= 7 + C \implies C = 1 \\ f'(x) &= 2x^4 + 5x + 1 \\ f(x) &= \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + \hat{C} \\ 0 &= f(1) \\ &= \frac{2}{5}(1)^5 + \frac{5}{2}(1)^2 + (1) + \hat{C} \\ &= \frac{39}{10} + \hat{C} \implies \hat{C} = -\frac{39}{10} \\ f(x) &= \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10} \end{aligned}$$

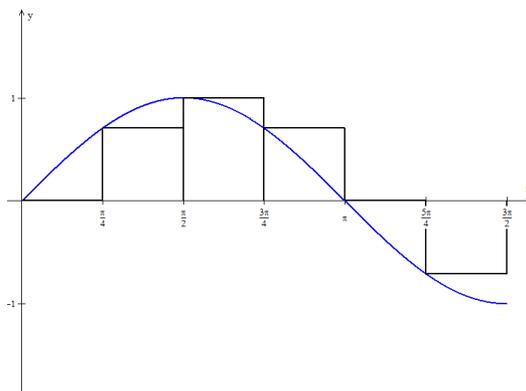
5. A particle is moving so that $a(t) = 3 \cos t - 2 \sin t$ with $s(0) = 0$ and $v(0) = 4$. Find the position of the particle.

Solution:

$$\begin{aligned}
 v(t) &= 3 \sin t + 2 \cos t + C \\
 4 &= v(0) \\
 &= 3 \sin 0 + 2 \cos 0 + C \\
 &= 2 + C \implies C = 2 \\
 v(t) &= 3 \sin t + 2 \cos t + 2 \\
 s(t) &= -3 \cos t + 2 \sin t + 2t + \hat{C} \\
 0 &= s(0) \\
 &= -3 \cos 0 + 2 \sin 0 + 2(0) + \hat{C} \\
 &= -3 + \hat{C} \implies \hat{C} = 3 \\
 s(t) &= -3 \cos t + 2 \sin t + 2t + 3
 \end{aligned}$$

6. Write a Riemann sum for $f(x) = \sin x$ on $0 \leq x \leq \frac{3\pi}{2}$ with six subintervals, taking sample points to be left endpoints, then find the sum.

Solution: Begin by graphing $y = \sin x$. Split the x -axis into six subintervals on $[0, \frac{3\pi}{2}]$. So $\Delta x = \frac{\frac{3\pi}{2} - 0}{6} = \frac{\pi}{4}$. Draw rectangles on each of those subintervals, taking heights to be vertical from the left endpoints.

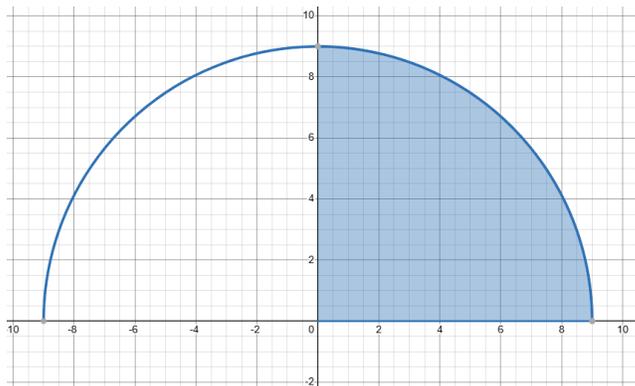


Now, $n = 6$, $f(x) = \sin x$, and $x_i^* = a + i\Delta x = 0 + i\frac{\pi}{4}$, so the Riemann sum is

$$\begin{aligned}
 L_6 &= \sum_{i=1}^n f(x_i^*) \Delta x = \sum_{i=1}^6 \sin\left(i\frac{\pi}{4}\right) \frac{\pi}{4} \\
 &= \frac{\pi}{4} \left(\sin 0 + \sin \frac{\pi}{4} + \sin \frac{\pi}{2} + \sin \frac{3\pi}{4} + \sin \pi + \sin \frac{5\pi}{4} \right) \\
 &= \frac{\pi}{4} \left(0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} \right) = \frac{\pi(\sqrt{2} + 2)}{8}
 \end{aligned}$$

7. Evaluate $\int_0^9 \sqrt{81 - x^2} dx$.

Solution: The graph of $y = \sqrt{81 - x^2}$ is a semicircle centered at the origin with radius 9.

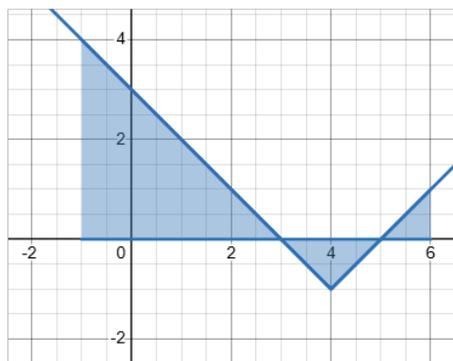


Thus, the definite integral represents a quarter of the full circle of radius 9. Therefore,

$$\int_0^9 \sqrt{81 - x^2} dx = \frac{1}{4} \pi (9)^2 = \frac{81\pi}{4}$$

8. Evaluate $\int_{-1}^6 (|x - 4| - 1) dx$.

Solution: The graph of $y = |x - 4| - 1$ is an absolute value graph shifted to the right 4 and down 1. We are looking for the area of the region between the x -axis and the curve.



This region is formed by 3 triangles. The triangle between $-1 \leq x \leq 3$ has height 4 and width 4, so its area is $\frac{1}{2}(4)(4) = 8$. The second triangle, between $3 \leq x \leq 5$, has height 1 and width 2, so its area is 1. Since it is beneath the x -axis, we will consider it negative. The third triangle has area $\frac{1}{2}$, so then

$$\int_{-1}^6 (|x - 4| - 1) dx = 8 - 1 + \frac{1}{2} = \frac{15}{2}$$

9. Estimate $\int_3^9 f(x) dx$ with three equal subintervals using

- (a) Right endpoints
- (b) Left endpoints

where values of $f(x)$ are given in the table below.

x	3	4	5	6	7	8	9
$f(x)$	-3.4	-2.1	-0.6	0.3	0.9	1.4	1.8

Solution: For all parts, $\Delta x = \frac{b-a}{n} = \frac{9-3}{3} = 2$. With three subintervals, the endpoints are $x_0 = 3, x_1 = 5, x_2 = 7, x_3 = 9$. Then $\int_3^9 f(x) dx \approx R_3, L_3, M_3$.

$$R_3 = 2(f(x_1) + f(x_2) + f(x_3)) = 2(-0.6 + 0.9 + 1.8) = 4.2$$

$$L_3 = 2(f(x_0) + f(x_1) + f(x_2)) = 2(-3.4 - 0.6 + 0.9) = -6.2$$

10. Evaluate $\int \sec(x) \tan(x) dx$.

Solution: By our table of integrals, we get $\int \sec(x) \tan(x) dx = \sec(x) + C$.

11. Evaluate $\int_0^{\frac{\pi}{6}} \sec(x) \tan(x) dx$.

Solution: Using FTC2, we get

$$\begin{aligned} \int_0^{\frac{\pi}{6}} \sec(x) \tan(x) dx &= [\sec(x)]_0^{\frac{\pi}{6}} \\ &= \sec\left(\frac{\pi}{6}\right) - \sec(0) \\ &= \frac{2\sqrt{3}}{3} - 1 \end{aligned}$$

12. Evaluate $\int 3x^2 e^{-x^3} dx$.

Solution: Let $u = -x^3$, so $du = -3x^2 dx$ and $-du = 3x^2 dx$. Then

$$\begin{aligned} \int 3x^2 e^{-x^3} dx &= -\int e^u du \\ &= -e^u + C \\ &= -e^{-x^3} + C \end{aligned}$$

13. Evaluate $\int_0^{\frac{\pi}{4}} \sin x \sin(\cos x) dx$.

Solution: Let $u = \cos x$, so $du = -\sin x dx$ and $-du = \sin x dx$. Then our upper and lower limits of integration are $u_U = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $u_L = \cos 0 = 1$, respectively. Then

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin x \sin(\cos x) dx &= -\int_1^{\frac{\sqrt{2}}{2}} \sin u du \\ &= \cos u \Big|_1^{\frac{\sqrt{2}}{2}} \\ &= \cos \frac{\sqrt{2}}{2} - \cos 1\end{aligned}$$

14. Evaluate $\int_{-31415926}^{31415926} \frac{x^5 \sin x \tan x |x|}{12 + x^2 + x^8} dx$.

Solution: Since $12 + x^2 + x^8$ and $|x|$ are even while x^5 , $\sin x$, and $\tan x$ are odd, our function is an odd function. Therefore,

$$\int_{-31415926}^{31415926} \frac{x^5 \sin x \tan x |x|}{12 + x^2 + x^8} dx = 0$$

15. Evaluate $\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}} dx$.

Solution: Let $u = 1 + (\ln x)^2$, $du = \frac{2 \ln x}{x} dx$, and $\frac{1}{2} du = \frac{\ln x}{x} dx$. Then

$$\begin{aligned}\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}} dx &= \frac{1}{2} \int \frac{1}{\sqrt{u}} du \\ &= u^{\frac{1}{2}} + C \\ &= \sqrt{(\ln x)^2 + 1} + C\end{aligned}$$

16. Evaluate $\int \frac{3t^2 - 2}{t^3 - 2t - 8} dt$.

Solution: Let $u = t^3 - 2t - 8$. Then $du = (3t^2 - 2) dt$. Then

$$\begin{aligned}\int \frac{3t^2 - 2}{t^3 - 2t - 8} dt &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|t^3 - 2t - 8| + C\end{aligned}$$

17. Evaluate $\frac{d}{dx} \int_0^x \sqrt[5]{t^2 + t + 1} dt$.

Solution: To evaluate the derivative of an accumulation function, we use FTC1. This says

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

For this prompt, $f(t) = \sqrt[5]{t^2 + t + 1}$, so $f(x) = \sqrt[5]{x^2 + x + 1}$. Thus,

$$\frac{d}{dx} \int_0^x \sqrt[5]{t^2 + t + 1} dt = \sqrt[5]{x^2 + x + 1}$$

18. Evaluate $\frac{d}{dx} \int_0^{\sin x} \sqrt[5]{t^2 + t + 1} dt$.

Solution: To evaluate the derivative of an accumulation function, we use FTC1. This says

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

For this prompt, $f(t) = \sqrt[5]{t^2 + t + 1}$, so $f(\sin(x)) = \sqrt[5]{\sin^2 x + \sin x + 1}$. We will also need to use the chain rule, since our upper limit of integration is a function of x . Thus,

$$\frac{d}{dx} \int_0^{\sin x} \sqrt[5]{t^2 + t + 1} dt = \sqrt[5]{\sin^2 x + \sin x + 1} \frac{d}{dx}(\sin x) = \sqrt[5]{\sin^2 x + \sin x + 1}(\cos x)$$