L'HÔPITAL'S RULE

MTH 253 LECTURE NOTES

We've spent time in the past with limits, and we've even studied a helpful strategy in L'Hôpital's Rule.

Example 1. Evaluate $\lim_{x\to 1} \dfrac{x^2-1}{x^2+x-2}$ both with and without L'Hôpital's Rule.

In order to use L'Hôpital's Rule, we need either an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Definition

In general, if we have $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$, then this limit may or may not exist and is called an **indeterminate form of type** $\frac{0}{0}$.
If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is called an **indeterminate form of type** $\frac{\infty}{\infty}$.

L'Hôspital's Rule:

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a). If $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit exists or is $\pm\infty$.

Example 2. Evaluate $\lim_{t\to \frac{\pi}{2}^-} (\sec x - \tan x)$.

The previous limit was transformed into an indeterminate form of type $\frac{0}{0}$ in order to evaluate it, but it was actually already an indeterminate form.

Definition

In general, the following limits may or may not exist and are all **indeterminate forms**.

- If $f(x) \to 0$ and $g(x) \to \pm\infty$ as $x \to a$, we call $\lim\limits_{x\to a} f(x)g(x)$ an **indeterminate form of type** $0 \cdot \infty$.
- If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, we call $\lim\limits_{x\to a}(f(x) - g(x))$ an **indeterminate form of type** $\infty - \infty$.
- If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, we call $\lim\limits_{x\to a} f(x)^{g(x)}$ an **indeterminate form of type** 0^0.
- If $f(x) \to \pm\infty$ and $g(x) \to 0$ as $x \to a$, we call $\lim\limits_{x\to a} f(x)^{g(x)}$ an **indeterminate form of type** ∞^0.
- If $f(x) \to 1$ and $g(x) \to \pm\infty$ as $x \to a$, we call $\lim\limits_{x\to a} f(x)^{g(x)}$ an **indeterminate form of type** 1^∞.

Example 3. Evaluate $\lim\limits_{x\to 0^+} x \ln x$.

Example 4. Evaluate $\lim\limits_{x\to\infty} \sqrt{x}^{e^{-x}}$.

Example 5. Evaluate $\lim_{x\to\pi}(\sin x)^{\tan x}$.

Example 6. Evaluate $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.