MTH 253Z
Final Review Key

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1. Determine whether the following series converges or diverges. If it converges, find its sum.

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

Solution: We can rewrite our series in sigma notation as

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

We can manipulate our series now as

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = 9\sum_{n=1}^{\infty} \frac{1}{10^n} = 9\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$$

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ is a geometric series with $r = \frac{1}{10} \in (-1, 1)$, so it is convergent. Moreover,

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n &= \frac{a}{1-r} \\ &= \frac{\frac{1}{10}}{1-\frac{1}{10}} \\ &= \frac{1}{9}\end{aligned}$$

It follows that $\displaystyle\sum_{n=1}^{\infty} \frac{9}{10^n} = 9\left(\frac{1}{9}\right) = 1.$

2. Determine whether the following series converges or diverges. If it converges, find its sum. Justify your conclusion as specifically as possible.

$$686 + 588 + 504 + 432 + \cdots$$

> **Solution:** Notice that each term of the series is obtained from the previous term by multiplying by $r = \frac{6}{7}$. Thus, our series is a convergent geometric series with $r \in (-1, 1)$. It follows that our series converges to $\dfrac{a}{1-r} = \dfrac{686}{1-\frac{6}{7}} = 4802$.

3. Determine whether the following series converges or diverges. If it converges, find its sum. Justify your conclusion as specifically as possible.

$$432 + 504 + 588 + 686 + \cdots$$

> **Solution:** Notice that each term of the series is obtained from the previous term by multiplying by $r = \frac{7}{6}$. Thus, our series is a divergent geometric series with $r \notin (-1, 1)$.

4. Determine whether the following series converges or diverges. If it converges, find its sum. Justify your conclusion as specifically as possible. *Hint: Consider a partial fraction decomposition.*

$$\sum_{n=0}^{\infty} \frac{2}{(n+1)(n+2)}$$

> **Solution:** Let's find the partial fraction decomposition of $\dfrac{2}{(n+1)(n+2)}$. It turns out,
>
> $$\frac{2}{(n+1)(n+2)} = \frac{2}{n+1} - \frac{2}{n+2}$$
>
> Then the nth partial sum of $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(n+1)(n+2)} = \sum_{n=0}^{\infty} \left(\frac{2}{n+1} - \frac{2}{n+2} \right)$ is
>
> $$\begin{aligned} s_n &= \frac{2}{1} - \frac{2}{2} + \frac{2}{2} - \frac{2}{3} + \frac{2}{3} - \frac{2}{4} + \cdots + \frac{2}{n+1} - \frac{2}{n+2} \\ &= 2 - \frac{2}{n+2} \end{aligned}$$
>
> Since $\displaystyle\lim_{n\to\infty} s_n = 2$, it follows that $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(n+1)(n+2)}$ converges to 2.

5. Determine whether the following series converges or diverges. Justify your conclusion as specifically as possible.

$$\sum_{n=4}^{\infty} \frac{7}{n^{\frac{6}{7}}}$$

Solution: Note that $\displaystyle\sum_{n=4}^{\infty} \frac{7}{n^{\frac{6}{7}}} = 7\sum_{n=4}^{\infty} \frac{1}{n^{\frac{6}{7}}}$. Since $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{6}{7}}}$ is a p-series with $p = \frac{6}{7} <= 1$, so $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\frac{6}{7}}}$ diverges. Therefore, $\displaystyle\sum_{n=4}^{\infty} \frac{7}{n^{\frac{6}{7}}}$ diverges.

6. Determine whether the following series converges or diverges. Justify your conclusion as specifically as possible.

$$\sum_{n=4}^{\infty} \frac{e}{n^{\pi}}$$

Solution: Note that $\displaystyle\sum_{n=4}^{\infty} \frac{e}{n^{\pi}} = e\sum_{n=4}^{\infty} \frac{1}{n^{\pi}}$. Since $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^{\pi}}$ converges as a p-series with $p = \pi > 1$, the original series $\displaystyle\sum_{n=4}^{\infty} \frac{e}{n^{\pi}}$ converges.

7. Determine whether the following series converges or diverges. Justify your conclusion as specifically as possible.

$$\sum_{n=1}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^3+n^2+n+1}$$

Solution: Expanding the summand, we get

$$\frac{(n+1)(2n+1)(3n+1)}{n^3+n^2+n+1} = \frac{6n^3+11n^2+6n+1}{n^3+n^2+n+1}.$$

Consider

$$\begin{aligned}\lim_{n\to\infty} \frac{(n+1)(2n+1)(3n+1)}{n^3+n^2+n+1} &= \lim_{n\to\infty} \frac{6n^3+11n^2+6n+1}{n^3+n^2+n+1} \\ &= 6\end{aligned}$$

By the Divergence Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^3+n^2+n+1}$ diverges.

8. Determine whether the following series converges or diverges. Justify your conclusion as specifically as possible.

$$\sum_{n=2}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^4 - n^3 - n^2 - n - 1}$$

Solution: Expanding the summand, we get

$$\frac{(n+1)(2n+1)(3n+1)}{n^3 + n^2 + n + 1} = \frac{6n^3 + 11n^2 + 6n + 1}{n^4 - n^3 - n^2 - n - 1}.$$

Notice that

$$\frac{6n^3 + 11n^2 + 6n + 1}{n^4 - n^3 - n^2 - n - 1} \geq \frac{6n^3}{n^4} \geq \frac{n^3}{n^4} = \frac{1}{n}$$

for all n. Because $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, both as part of the harmonic series and a p-series with $p = 1 \leq 1$, it must be the case that $\sum_{n=2}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^4 - n^3 - n^2 - n - 1}$ diverges by the comparison test.

9. Determine whether the following series converges or diverges. Justify your conclusion as specifically as possible.

$$\sum_{n=2}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^4+n^3+n^2+n+1}$$

Solution: Since both the numerator and denominator are both obtained by increasing the leading terms of the numerator and denominator, we cannot justly compare the two summands

$$\frac{(n+1)(2n+1)(3n+1)}{n^4+n^3+n^2+n+1} \quad \text{and} \quad \frac{6n^3}{n^4}$$

with an inequality. However, we suppose that $\frac{(n+1)(2n+1)(3n+1)}{n^4+n^3+n^2+n+1}$ is similar to $\frac{6n^3}{n^4}$. Let's use the Limit Comparison Test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{(n+1)(2n+1)(3n+1)}{n^4+n^3+n^2+n+1}}{\frac{6n^3}{n^4}} &= \lim_{n\to\infty} \frac{(n+1)(2n+1)(3n+1)}{n^4+n^3+n^2+n+1} \cdot \frac{n^4}{6n^3} \\
&= \lim_{n\to\infty} \frac{6n^3+11n^2+6n+1}{n^4+n^3+n^2+n+1} \cdot \frac{n^4}{6n^3} \\
&= \lim_{n\to\infty} \frac{6n^7+11n^6+6n^5+n^4}{6n^7+6n^6+6n^5+6n^4+6n^3} \\
&= \lim_{n\to\infty} \frac{6+\frac{11}{n}+\frac{6}{n^2}+\frac{1}{n^3}}{6+\frac{6}{n}+\frac{6}{n^2}+\frac{6}{n^3}+\frac{6}{n^4}} \\
&= \frac{6}{6} \\
&= 1
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{\frac{(n+1)(2n+1)(3n+1)}{n^4+n^3+n^2+n+1}}{\frac{6n^3}{n^4}} = 1$, either both $\sum_{n=2}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^4+n^3+n^2+n+1}$ and $\sum_{n=2}^{\infty} \frac{6n^3}{n^4}$ both converge or both diverge. Because $\sum_{n=2}^{\infty} \frac{6n^3}{n^4} = 6\sum_{n=2}^{\infty} \frac{1}{n}$ diverges as part of the harmonic series or as a p-series with $p = 1 \leq 1$, it must be the case that $\sum_{n=2}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^4+n^3+n^2+n+1}$ diverges by the Limit Comparison Test.

10. Determine whether the following series converges or diverges. Justify your conclusion as specifically as possible.

$$\sum_{n=1}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^5+n^4+n^3+n^2+n+1}$$

Solution: Since both the numerator and denominator are both obtained by increasing the leading terms of the numerator and denominator, we cannot justly compare the two summands

$$\frac{(n+1)(2n+1)(3n+1)}{n^5+n^4+n^3+n^2+n+1} \qquad \text{and} \qquad \frac{6n^3}{n^4}$$

with an inequality. However, we suppose that $\frac{(n+1)(2n+1)(3n+1)}{n^5+n^4+n^3+n^2+n+1}$ is similar to $\frac{6n^3}{n^5}$. Let's use the Limit Comparison Test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{(n+1)(2n+1)(3n+1)}{n^5+n^4+n^3+n^2+n+1}}{\frac{6n^3}{n^5}} &= \lim_{n\to\infty} \frac{(n+1)(2n+1)(3n+1)}{n^5+n^4+n^3+n^2+n+1} \cdot \frac{n^5}{6n^3} \\
&= \lim_{n\to\infty} \frac{6n^3+11n^2+6n+1}{n^5+n^4+n^3+n^2+n+1} \cdot \frac{n^5}{6n^3} \\
&= \lim_{n\to\infty} \frac{6n^8+11n^7+6n^6+n^5}{6n^8+6n^7+6n^6+6n^5+6n^4+6n^3} \\
&= \lim_{n\to\infty} \frac{6+\frac{11}{n}+\frac{6}{n^2}+\frac{1}{n^3}}{6+\frac{6}{n}+\frac{6}{n^2}+\frac{6}{n^3}+\frac{6}{n^4}+\frac{6}{n^3}} \\
&= \frac{6}{6} \\
&= 1
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{\frac{(n+1)(2n+1)(3n+1)}{n^5+n^4+n^3+n^2+n+1}}{\frac{6n^3}{n^5}} = 1$, either both $\sum_{n=2}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^5+n^4+n^3+n^2+n+1}$ and $\sum_{n=2}^{\infty} \frac{6n^3}{n^5}$ both converge or both diverge. Because $\sum_{n=2}^{\infty} \frac{6n^3}{n^5} = 6\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p = 2 > 1$, it must be the case that $\sum_{n=2}^{\infty} \frac{(n+1)(2n+1)(3n+1)}{n^5+n^4+n^3+n^2+n+1}$ converges by the Limit Comparison Test.

11. Determine whether the following series converges or diverges. Justify your conclusion as specifically as possible.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{n^4+1}$$

Solution: Let $b_n = \dfrac{n^3}{n^4+1}$. Now, b_n consists of positive terms, b_n is decreasing (it is a rational function with denominator being a larger power), and $\lim_{n\to\infty} b_n = 0$. Since $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ is an alternating series, it converges by the Alternating Series Test.

12. Determine whether the following series converges or diverges. Justify your conclusion as specifically as possible.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5^{3n}}{\sqrt[3]{n}}$$

Solution: Note that $\lim_{n\to\infty} \dfrac{(-1)^{n+1}5^{3n}}{\sqrt[3]{n}}$ DNE. It follows that $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}5^{3n}}{\sqrt[3]{n}}$ diverges by the Divergence Test.

(Note: The Ratio Test works well with this problem, too)

13. Express the function $f(x) = \dfrac{x^2}{2+x^3}$ as a the sum of a power series and find its radius of convergence.

Solution: It seems as though $\dfrac{x^2}{2+x^3}$ is somewhat similar to $\dfrac{1}{1-\square}$. Now,

$$\begin{aligned}\frac{x^2}{2+x^3} &= x^2\frac{1}{2+x^3}\\ &= x^2\frac{1}{2\left(1+\frac{x^3}{2}\right)}\\ &= \frac{x^2}{2}\cdot\frac{1}{1-\left(\frac{-x^3}{2}\right)}\end{aligned}$$

Because $\dfrac{1}{1-\square} = \sum_{n=0}^{\infty}\square^n$, it follows that $\frac{1}{1-\left(\frac{-x^3}{2}\right)} = \sum_{n=0}^{\infty}\left(\dfrac{-x^3}{2}\right)^n$. Therefore,

$$\begin{aligned}f(x) &= \frac{x^2}{2+x^3}\\ &= \frac{x^2}{2}\cdot\frac{1}{1-\left(\frac{-x^3}{2}\right)}\\ &= \frac{x^2}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{-x^3}{2}\right)^n\\ &= \frac{x^2}{2}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{2^n}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+2}}{2^{n+1}}\end{aligned}$$

Since $\sum_{n=0}^{\infty}\square^n$ converges when $|\square| < 1$, it follows that $\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{3n+2}}{2^{n+1}}$ converges when $\left|\frac{-x^3}{2}\right| < 1$. Now,

$$\begin{aligned}\left|\frac{-x^3}{2}\right| &< 1\\ |x^3| &< 2\\ |x| &< \sqrt[3]{2}\end{aligned}$$

It follows that the radius of convergence for our power series is $R = \sqrt[3]{2}$.

14. Find the Taylor series for $f(x) = \left(\frac{2}{3}\right)^x$ centered at 1 and find its interval of convergence.

Solution: Taylor series have the form

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots$$

We need to calculate $f^{(n)}(a)$ for several values of n in order to find a pattern for the Taylor series.

$f(x)$	$=$	$\left(\frac{2}{3}\right)^x$
$f'(x)$	$=$	$\ln\frac{2}{3}\left(\frac{2}{3}\right)^x$
$f''(x)$	$=$	$\left(\ln\frac{2}{3}\right)^2\left(\frac{2}{3}\right)^x$
$f'''(x)$	$=$	$\left(\ln\frac{2}{3}\right)^3\left(\frac{2}{3}\right)^x$
$f^{(4)}(x)$	$=$	$\left(\ln\frac{2}{3}\right)^4\left(\frac{2}{3}\right)^x$
$\vdots$		$\vdots$
$f^{(n)}(x)$	$=$	$\left(\ln\frac{2}{3}\right)^n\left(\frac{2}{3}\right)^x$

$f(1)$	$=$	$\frac{2}{3}$
$f'(1)$	$=$	$\frac{2}{3}\ln\frac{2}{3}$
$f''(1)$	$=$	$\frac{2}{3}\left(\ln\frac{2}{3}\right)^2$
$f'''(1)$	$=$	$\frac{2}{3}\left(\ln\frac{2}{3}\right)^3$
$f^{(4)}(1)$	$=$	$\frac{2}{3}\left(\ln\frac{2}{3}\right)^4$
$\vdots$		$\vdots$
$f^{(n)}(1)$	$=$	$\frac{2}{3}\left(\ln\frac{2}{3}\right)^n$

It follows that the Taylor series for $f(x)$ at $x = 1$ is

$$\sum_{n=0}^{\infty} \frac{2\left(\ln\frac{2}{3}\right)^n}{3 \cdot n!}(x-1)^n$$

Its interval of convergence can be found by using the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2\left(\ln\frac{2}{3}\right)^{n+1}(x-1)^{n+1}}{3\cdot(n+1)!}\cdot\frac{3\cdot n!}{2\left(\ln\frac{2}{3}\right)^n(x-1)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{\ln\frac{2}{3}(x-1)}{n+1}\right| \\
&= 0
\end{aligned}$$

It follows that the interval of convergence is $\mathbb{R}$.

15. Use the binomial series to find the series expansion of $\frac{-2}{\sqrt[4]{32+2x}}$.

Solution: From the binomial series, $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. We need to make $\frac{-2}{\sqrt[4]{32+2x}}$ have this form. Thus,

$$\begin{aligned}
\frac{-2}{\sqrt[4]{32+2x}} &= -2\frac{1}{\sqrt[4]{2(16+x)}} \\
&= -2\frac{1}{\sqrt[4]{2}\sqrt[4]{1+\frac{x}{16}}} \\
&= \frac{-2}{\sqrt[4]{2}}\frac{1}{\sqrt[4]{1+\frac{x}{16}}} \\
&= -\sqrt[4]{2^3}\frac{1}{\sqrt[4]{1+\frac{x}{16}}} \\
&= -\sqrt[4]{2^3}\left(1+\frac{x}{16}\right)^{\frac{-1}{4}}
\end{aligned}$$

From the binomial series, $\left(1+\frac{x}{16}\right)^{\frac{-1}{4}} = \sum_{n=0}^{\infty} \binom{\frac{-1}{4}}{n} \left(\frac{x}{16}\right)^n$. It follows that

$$\frac{-2}{\sqrt[4]{32+2x}} = -\sqrt[4]{2^3}\left(1+\frac{x}{16}\right)^{\frac{-1}{4}} = -\sqrt[4]{2^3}\sum_{n=0}^{\infty} \binom{\frac{-1}{4}}{n} \left(\frac{x}{16}\right)^n$$

16. Use the binomial series to find the coefficient of the third-degree term in the series expansion of $\frac{-2}{\sqrt[4]{32+2x}}$.

Solution: From the previous problem, $\frac{-2}{\sqrt[4]{32+2x}} = -\sqrt[4]{2^3}\sum_{n=0}^{\infty} \binom{\frac{-1}{4}}{n} \left(\frac{x}{16}\right)^n$. Now, the third-degree term in this expansion is

$$\begin{aligned}
-\sqrt[4]{2^3}\binom{\frac{-1}{4}}{3}\left(\frac{x}{16}\right)^3 &= -\sqrt[4]{2^3}\frac{\left(\frac{-1}{4}\right)\left(\frac{-5}{4}\right)\left(\frac{-9}{4}\right)}{3!}\frac{x^3}{16^3} \\
&= -2^{\frac{3}{4}}\frac{(-1)(-5)(-9)}{4\cdot 4\cdot 4\cdot 3\cdot 2\cdot 1}\frac{x^3}{16^3} \\
&= 2^{\frac{3}{4}}\frac{45}{2^{19}\cdot 3}x^3 \\
&= \frac{15}{2^{\frac{73}{4}}}x^3
\end{aligned}$$

It follows that the coefficient for the third-degree term is $\frac{15}{2^{\frac{73}{4}}}$.

17. Estimate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{n^4+1}$ with the fourth partial sum of the series.

Solution:

$$\begin{aligned} s_4 &= \sum_{n=1}^{4} \frac{(-1)^{n+1} n^3}{n^4+1} \\ &= \frac{1^3}{1^4+1} - \frac{2^3}{2^4+1} + \frac{3^3}{3^4+1} - \frac{4^3}{4^4+1} \\ &= \frac{19642}{179129} \end{aligned}$$

18. How many terms must be used to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{n^4+1}$ to within 0.0001 of the true value of the sum? Use Desmos to support your conclusion.

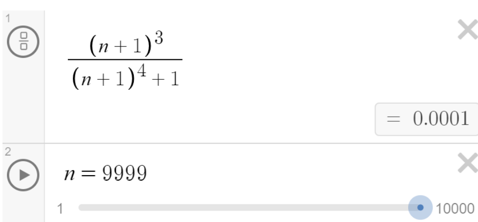
Solution: This is an alternating series with $b_n = \frac{n^3}{n^4+1}$. Note that b_n is positive, decreasing, and $\lim_{n\to\infty} b_n = 0$, so we can use the Alternating Series Estimation Theorem.

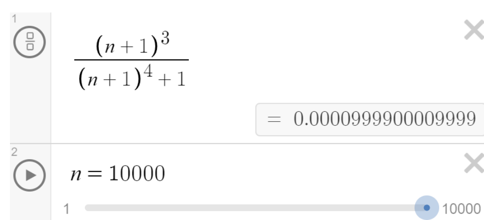
By the Alternating Series Estimation Theorem, $|R_n| \le b_{n+1}$. So when $b_{n+1} \le 0.0001$, we would have $|R_n| \le 0.0001$.

Consider,

$$\begin{aligned} b_{n+1} &\le 0.0001 \\ \frac{(n+1)^3}{(n+1)^4+1} &\le 0.0001 \end{aligned}$$

Algebraically, this is a *very* difficult problem. Using Desmos, we see that the first value of n that satisfies this inequality is $n = 10000$.





It follows that we need 10000 terms in order to ensure that our approximation is within 0.0001 of the true value of the sum.