MTH 254 Final Review Key

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1. Evaluate the limit.

$$\lim_{(x,y)\to(2,2)} \frac{e^x + \ln(y-1)\cos(\pi xy)}{\sqrt[5]{x^2y^3} + xy}$$

Solution: This function is continuous on its domain, so let's try evaluating at $(2,2)$.

$$\begin{aligned}
\lim_{(x,y)\to(2,2)} \frac{e^x + \ln(y-1)\cos(\pi xy)}{\sqrt[5]{x^2y^3} + xy} &= \frac{e^2 + \ln(2-1)\cos(\pi(2)(2))}{\sqrt[5]{2^2 2^3} + (2)(2)} \\
&= \frac{e^2 + \ln(1)\cos(4\pi)}{\sqrt[5]{2^5} + 4} \\
&= \frac{e^2 + (0)(1)}{2+4} \\
&= \frac{e^2}{6}
\end{aligned}$$

2. Suppose f is a differentiable function of two variables.
   (a) Explain as specifically as possible what $f_y(a,b) = 1$ means.
   (b) Explain as specifically as possible what $f_x(1,1) = -3$ means.
   (c) Explain as specifically as possible what $D_{\mathbf{i}}f(a,b) = 2$ means.
   (d) Explain as specifically as possible what $D_{\mathbf{u}}f(1,1) = -3$ means. For this to be well-defined, what property must $\mathbf{u}$ have?

Solution:

(a) $f_y(a,b)$ represents the instantaneous change of f in the y-direction at the point (a,b). Since $f_y(a,b) = 1$, this means that as we move from (a,b) in the positive y direction, we anticipate a gain of 1 unit of z as we move 1 unit in the y-direction.

(b) $f_x(1,1) = -3$ means that the slope on the surface $z = f(x,y)$ at $(1,1)$ is -1 while looking in the positive x direction.

(c) $D_{\mathbf{i}}f(a,b)$ represents the instantaneous change of f at (a,b) in the direction of $\mathbf{i}$ (the positive x direction).

(d) $D_{\mathbf{u}}f(1,1) = -3$ means that if I were to stand at the point $(1,1,f(1,1))$ and look in the direction of $\mathbf{u}$, I would see a slope that falls downward -3 as I move 1 in the direction of $\mathbf{u}$. For this to be well-defined, $\mathbf{u}$ must be a unit vector.

3. Below are some statements. Determine if the statement is True or False. If the statement is True, you need only write "True" and do not need to provide a justification. If the statement is False, write "False" and justify your answer as specifically as possible.
   (a) Suppose f is a differentiable two-variable function. Then $D_{\mathbf{i}}f(x,y) = f_x(x,y)$.
   (b) Suppose f is a differentiable three-variable function. Then $\dfrac{\partial^3 f}{\partial x \partial y \partial z} = f_{xyz}$.
   (c) Suppose f is a differentiable two-variable function. Then $(a,b) \in \mathbb{R}^2$, $f_{xy}(a,b) = f_{yx}(a,b)$.
   (d) For all (a,b) in the domain of a two-variable function f, $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.
   (e) $\displaystyle\int_{-1}^{2}\int_{0}^{6} x^2 \sin(x-y)\; dx\; dy = \int_{0}^{6}\int_{-1}^{2} x^2 \sin(x-y)\; dy\; dx.$

Solution:
(a) True.
(b) False. The order of differentiation is different for each expression.
(c) False. This is only true when f has continuous second-order partial derivatives.
(d) False. This is only true if f is continuous at (a,b).
(e) True.

4. Let $z = f(x,y)$, where $f(x,y) = 1 + x + y^2 - \sin(xy) + \ln(x^4 + y^2)$, and let S be the surfaced produced by that equation.
   (a) Find f_{xx}.
   (b) Find f_{xy}.
   (c) Find f_{yyx}.
   (d) Find the linear approximation to S at the point $(1,0,2)$.

Solution:
(a) Now

$$\begin{aligned}
f_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(f)\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(1 + x + y^2 - \sin(xy) + \ln(x^4 + y^2))\right) \\
&= \frac{\partial}{\partial x}\left(1 - y\cos(xy) + \frac{4x^3}{x^4 + y^2}\right) \\
&= y^2\cos(xy) + \frac{12x^2(x^4 + y^2) - 16x^6}{(x^4 + y^2)^2}
\end{aligned}$$

(b) Now

$$\begin{aligned}
f_{xy} &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(f)\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(1 + x + y^2 - \sin(xy) + \ln(x^4 + y^2))\right) \\
&= \frac{\partial}{\partial y}\left(1 - y\cos(xy) + \frac{4x^3}{x^4 + y^2}\right) \\
&= -\cos(xy) + xy\sin(xy) - \frac{8x^3 y}{(x^4 + y^2)^2}
\end{aligned}$$

(c) Now

$$\begin{aligned}
f_{yyx} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(f)\right)\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(1 + x + y^2 - \sin(xy) + \ln(x^4 + y^2))\right)\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(2y - x\cos(xy) + \frac{2y}{x^4 + y^2}\right)\right) \\
&= \frac{\partial}{\partial x}\left(2 + x^2\sin(xy) + \frac{2(x^4 + y^2) - 4y^2}{(x^4 + y^2)^2}\right) \\
&= \frac{\partial}{\partial x}\left(2 + x^2\sin(xy) + \frac{2x^4 - 2y^2}{(x^4 + y^2)^2}\right) \\
&= 2x\sin(xy) + x^2y\cos(xy) + \frac{8x^3(x^4 + y^2)^2 - 8x^3(2x^4 - 2y^2)(x^4 + y^2)}{(x^4 + y^2)^4} \\
&= 2x\sin(xy) + x^2y\cos(xy) + \frac{8x^3(x^4 + y^2) - 8x^3(2x^4 - 2y^2)}{(x^4 + y^2)^3} \\
&= 2x\sin(xy) + x^2y\cos(xy) + \frac{-8x^7 + 24x^3y^2}{(x^4 + y^2)^3}
\end{aligned}$$

(d) The linear approximation to S at $(1, 0, 2)$ is

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

Then

$$\begin{aligned}
f(1, 0) &= 2 \\
f_x(x, y) &= 1 - y\cos(xy) + \frac{4x^3}{x^4 + y^2} \\
f_x(1, 0) &= 1 - 0\cos(0) + \frac{4(1)^3}{(1)^4 + (0)^2} \\
&= 5 \\
f_y(x, y) &= 2y - x\cos(xy) + \frac{2y}{x^4 + y^2} \\
f_y(1, 0) &= 2(0) - (1)\cos(0) + \frac{2(0)}{(1)^4 + (0)^2} \\
&= -1
\end{aligned}$$

It follows that the linear approximation $L(x, y) = 2 + 5(x - 1) - y = 5x - y - 3$.

5. Find the linear equation of the plane tangent to the surface whose equation is $z = e^x \cos y$ at $(0, 0, 1)$.

Solution: An equation for the plane tangent to the surface whose equation is $z = f(x, y)$ at (a, b, c) is

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - c) = 0$$

Solving for z yields the linear equation, which is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

It follows that

$$\begin{aligned} f_x(x, y) &= e^x \cos y \\ f_x(0, 0) &= e^0 \cos 0 = 1 \\ f_y(x, y) &= -e^x \sin y \\ f_y(0, 0) &= -e^0 \sin 0 = 0 \end{aligned}$$

Then the linear equation fort he tangent plane is $z = 1 + 1(x - 0) + 0(y - 0)$, so $z = 1 + x$.

6. Find the symmetric equations for the normal line to the surface $\sin(xyz) = x + 2y + 3z$ at the point $(2, -1, 0)$.

Solution: The symmetric equations for a line are $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, where $\langle a, b, c\rangle$ is the direction vector of the line. We find the direction vector by moving all terms to one side and finding the gradient. That is,

$$\begin{aligned} f(x, y, z) &= -\sin(xyz) + x + 2y + 3z \\ \nabla f(x, y, z) &= \langle -yz \cos(xyz) + 1, -xz \cos(xyz) + 2, -xy \cos(xyz) + 3\rangle \\ \nabla f(2, -1, 0) &= \langle -(-1)(0) \cos((2)(-1)(0)) + 1, -(2)(0) \cos((2)(-1)(0)) + 2, -(2)(-1) \cos((2)(-1)(0)) + 3\rangle \\ &= \langle 1, 2, 5\rangle \end{aligned}$$

Thus, the symmetric equations are $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-0}{5}$, or

$$x - 1 = \frac{y + 1}{2} = \frac{z}{5}$$

7. If $\sin(xyz) = x + 2y + 3z$, find $\dfrac{\partial x}{\partial y}$.

Solution: To find $\frac{\partial x}{\partial y}$, we differentiate both sides with respect to y, treating x as a function of y. That is,

$$\begin{aligned} \frac{\partial}{\partial y}(\sin(xyz)) &= \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial y}(3z) \\ \cos(xyz)\frac{\partial}{\partial y}(xyz) &= \frac{\partial x}{\partial y} + 2 + 0 \\ \cos(xyz)\left(\frac{\partial x}{\partial y} yz + x(1)z + xy(0)\right) &= \frac{\partial x}{\partial y} + 2 \\ \frac{\partial x}{\partial y}\cos(xyz) + xy\cos(xyz) &= \frac{\partial x}{\partial y} + 2 \\ \frac{\partial x}{\partial y}(\cos(xyz) - 1) &= 2 - xy\cos(xyz) \\ \frac{\partial x}{\partial y} &= \frac{2 - xy\cos(xyz)}{\cos(xyz) - 1} \end{aligned}$$

8. Let $u = x^2y^3 + z^4$, with $x = p + 3p^2$, $y = pe^p$, and $z = p\sin p$. Find $\dfrac{du}{dp}$.

Solution: We notice that u is a function of x, y, and z, while x, y, and z are each functions of p. So then by the Chain Rule,

$$\frac{du}{dp} = \frac{\partial u}{\partial x}\frac{dx}{dp} + \frac{\partial u}{\partial y}\frac{dy}{dp} + \frac{\partial u}{\partial z}\frac{dz}{dp}$$

Let's compute each piece.

$$\begin{aligned} u &= x^2y^3 + z^4 \\ \frac{\partial}{\partial x}(u) &= \frac{\partial}{\partial x}(x^2y^3 + Z^4) \\ \frac{\partial u}{\partial x} &= 2xy^3 \\ \frac{\partial}{\partial y}(u) &= \frac{\partial}{\partial y}(x^2y^3 + Z^4) \\ \frac{\partial u}{\partial y} &= 3x^2y^2 \\ \frac{\partial}{\partial z}(u) &= \frac{\partial}{\partial z}(x^2y^3 + Z^4) \\ \frac{\partial u}{\partial z} &= 4z^3 \end{aligned}$$

$$\begin{aligned} x &= p + 3p^2 \\ \frac{dx}{dp} &= 1 + 6p \end{aligned}$$

$$\begin{aligned} y &= pe^p \\ \frac{dy}{dp} &= e^p + pe^p \end{aligned}$$

$$\begin{aligned} z &= p\sin p \\ \frac{dz}{dp} &= \sin p + p\cos p \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial u}{\partial p} &= \frac{\partial u}{\partial x}\frac{dx}{dp} + \frac{\partial u}{\partial y}\frac{dy}{dp} + \frac{\partial u}{\partial z}\frac{dz}{dp} \\ &= (2xy^3)(p + 3p^2) + (3x^2y^2)(e^p + pe^p) + (4z^3)(\sin p + p\cos p) \\ &= (2(p + 3p^2)(pe^p)^3)(p + 3p^2) + (3(p + 3p^2)^2(pe^p)^2)(e^p + pe^p) + (4(p\sin p)^3)(\sin p + p\cos p) \\ &= 45e^{3p}p^7 + 57e^{3p}p^6 + 23e^{3p}p^5 + 3e^{3p}p^4 + 4p^4\sin^3 p\cos p + 4p^3\sin^4 p \end{aligned}$$

9. Let $v = x^2 \sin y + ye^{xy}$, with $x = s + 2t$, and $y = st$. Find $\dfrac{\partial v}{\partial s}$ when $s = 0$ and $t = 1$.

Solution: By the Chain Rule, $\dfrac{\partial v}{\partial s} = \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial s}$. Let's compute.

$$\begin{aligned}
v &= x^2 \sin y + ye^{xy} \\
\frac{\partial}{\partial x}(v) &= \frac{\partial}{\partial x}(x^2 \sin y + ye^{xy}) \\
\frac{\partial v}{\partial x} &= 2x \sin y + y^2 e^{xy} \\
\frac{\partial}{\partial y}(v) &= \frac{\partial}{\partial y}(x^2 \sin y + ye^{xy}) \\
\frac{\partial v}{\partial y} &= x^2 \cos y + e^{xy} + xye^{xy}
\end{aligned}$$

$$\begin{aligned}
x &= s + 2t \\
\frac{\partial}{\partial s}(x) &= \frac{\partial}{\partial s}(s + 2t) \\
\frac{\partial x}{\partial s} &= 1
\end{aligned}$$

$$\begin{aligned}
y &= st \\
\frac{\partial}{\partial s}(y) &= \frac{\partial}{\partial s}(st) \\
\frac{\partial y}{\partial s} &= t
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v}{\partial s} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} \\
&= (2x \sin y + y^2 e^{xy})(1) + (x^2 \cos y + e^{xy} + xye^{xy})(2) \\
&= 2x \sin y + y^2 e^{xy} + 2x^2 \cos y + 2e^{xy} + 2xye^{xy} \\
&= 2(s + 2t)\sin(st) + (st)^2 e^{(s+2t)(st)} + 2(s + 2t)^2 \cos(st) + 2e^{(s+2t)(st)} \\
&\qquad + 2(s + 2t)(st)e^{(s+2t)(st)} \\
\left.\frac{\partial v}{\partial s}\right|_{s=0,t=1} &= 2((0) + 2(1))\sin((0)(1)) + ((0))^2 e^{((0)+2(1))((0)(1))} + 2((0) + 2(1))^2 \cos((0)(1)) \\
&\qquad + 2e^{((0)+2(1))((0)(1))} + 2((0) + 2(1))((0)(1))e^{((0)+2(1))((0)(1))} \\
&= 10
\end{aligned}$$

10. Let $F(x,y,z) = x^2 e^{yz^2}$, S be the level surface whose equation is $F(x,y,z) = 4$, and let P be the point $(2,0,4)$ on S.

   (a) Find ∇F.

   (b) Find a vector pointing in the direction for which $D_{\mathbf{u}}F(2,0,4)$ maximal.

   (c) Find the maximal value of $D_{\mathbf{u}}F(2,0,4)$.

   (d) Find an equation for the tangent plane to S at P.

   (e) Find a vector equation for the normal line to S at P.

Solution:

(a) $\nabla F = \langle F_x, F_y, F_z \rangle = \left\langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2} \right\rangle$.

(b) The directional derivative $D_{\mathbf{u}}F(2,0,4)$ has a maximum value of $|\nabla F(2,0,4)|$, and this occurs when $\mathbf{u}$ is parallel to $\langle 2,0,4 \rangle$. Thus, $D_{\mathbf{u}}F(2,0,4)$ is maximal in the direction of $\langle 2,0,4 \rangle$.

(c) The maximal value of $D_{\mathbf{u}}F(2,0,4)$ is

$$
\begin{aligned}
|\nabla F(2,0,4)| &= \left| \left\langle 2(2)e^{(0)(4)^2}, (2)^2(4)^2e^{(0)(4)^2}, 2(2)^2(0)(4)e^{(0)(4)^2} \right\rangle \right| \\
&= |\langle 4, 64, 0 \rangle| \\
&= \sqrt{4112} = 4\sqrt{257}
\end{aligned}
$$

(d) An equation for the tangent plane to S at P is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, where $\langle x_0, y_0, z_0 \rangle = \overrightarrow{OP}$, and $\langle a,b,c \rangle = \nabla F(P)$. Then

$$
\begin{aligned}
a(x-x_0) + b(y-y_0) + c(z-z_0) &= 0 \\
4(x-2) + 64(y-0) + 0(z-4) &= 0 \\
4x - 8 + 64y &= 0 \\
4x + 64y - 8 &= 0
\end{aligned}
$$

(e) The normal line to S at P is the line through the normal vector to the tangent plane to S at P. The normal vector for the tangent plane is $\mathbf{n} = \nabla F(2,0,4) = \langle 4, 64, 0 \rangle$. Thus, the vector equation for the normal line to S at P is

$$
\begin{aligned}
\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) &= 0 \\
\langle 4, 64, 0 \rangle \cdot \langle x-2, y, z-4 \rangle &= 0
\end{aligned}
$$

11. Let $f(x, y) = 3xy - x^2y - xy^2$. Find the local extrema and saddle points of f. Show all of your work to justify your conclusion.

Solution: To find the extrema and the saddle points, we can use the second derivative test and the discriminant. Recall that $D = f_{xx}f_{yy} - f_{xy}^2$.

To use the second derivative test, we begin by finding the critical points of f – this is where $f_x = 0$ and $f_y = 0$ (or where either of these is undefined). We compute

$$\begin{aligned} f_x &= 3y - 2xy - y^2 \\ f_y &= 3x - x^2 - 2xy \end{aligned}$$

$$\begin{aligned} 0 &= f_x \\ &= 3y - 2xy - y^2 \\ 2xy &= 3y - y^2 \end{aligned}$$

We can solve this for x, but that requires dividing by y. We may only divide by y if $y \neq 0$. We now have two cases, one where $y \neq 0$, and one where $y = 0$.

If $y \neq 0$, then

$$x = \frac{3-y}{2}$$

$$\begin{aligned} 0 &= f_y \\ 0 &= 3\left(\frac{3-y}{2}\right) - \left(\frac{3-y}{2}\right)^2 - 2\left(\frac{3-y}{2}\right)y \\ 0 &= \frac{9-3y}{2} - \frac{9-6y+y^2}{4} - 3y + y^2 \\ 0 &= 18 - 6y - 9 + 6y - y^2 - 12y + 4y^2 \\ 0 &= 9 - 12y + 3y^2 \\ 0 &= 3(y-1)(y-3) \\ y &= 1, 3 \end{aligned}$$

When we evaluate $x = \dfrac{3-y}{2}$ when $y = 1, 3$, we get the critical points $(1, 1)$ and $(0, 3)$.

Now, if $y = 0$, then

$$\begin{aligned} 0 &= f_y \\ &= 3x - x^2 - 2x(0) \\ &= x(3-x) \\ x &= 0, 3 \end{aligned}$$

So we have two more critical points of $(0, 0)$ and $(3, 0)$.

We now find the discriminant at each of our critical points.

$$\begin{aligned} f_x &= 3y - 2xy - y^2 \\ f_y &= 3x - x^2 - 2xy \end{aligned}$$

$$\begin{aligned} f_{xx} &= -2y \\ f_{xy} &= 3 - 2x - 2y \\ f_{yy} &= -2x \end{aligned}$$

$$
\begin{aligned}
D &= (-2y)(-2x) - (3 - 2x - 2y)^2 \\
&= -4x^2 - 4xy - 4y^2 + 12x + 12y - 9
\end{aligned}
$$

$$
\begin{aligned}
D(1,1) &= -4(1)^2 - 4(1)(1) - 4(1)^2 + 12(1) + 12(1) - 9 \\
&= 3 \\
D(0,3) &= -4(0)^2 - 4(0)(3) - 4(3)^2 + 12(0) + 12(3) - 9 \\
&= -9 \\
D(0,0) &= -4(0)^2 - 4(0)(0) - 4(0)^2 + 12(0) + 12(0) - 9 \\
&= -9 \\
D(3,0) &= -4(3)^2 - 4(3)(0) - 4(0)^2 + 12(3) + 12(0) - 9 \\
&= -9
\end{aligned}
$$

Since the discriminant at each of $(0,3),(0,0),(3,0)$ is negative, each of these correspond to saddle points.

As for $(1,1)$, we now need to compute $f_{xx}(1,1)$. If this is positive, we will have a local minimum. If this is negative, we will have a local maximum. Since $f_{xx}(1,1) = -2(1) = -2$, we have a local maximum at $(1,1)$.

Lastly, we need the z-values for each of these points.

$$
\begin{aligned}
f(x,y) &= 3xy - x^2y - xy^2 \\
f(0,3) &= 3(0)(3) - (0)^2(3) - (0)(3)^2 &&= 0 \\
f(0,0) &= 3(0)(0) - (0)^2(0) - (0)(0)^2 &&= 0 \\
f(3,0) &= 3(3)(0) - (3)^2(0) - (3)(0)^2 &&= 0 \\
f(1,1) &= 3(1)(1) - (1)^2(1) - (1)(1)^2 &&= 1
\end{aligned}
$$

It follows that f has saddle points at $(0,3,0),(0,0,0),(3,0,0)$. Moreover, f has a local maximum at $(1,1,1)$.

12. Use Lagrange multipliers to find the absolute extrema of $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$ subject to the constraint $x^2 + y^2 = 4$.

Solution: To find the absolute extrema of f subject to $x^2 + y^2 = 4$ using Lagrange multipliers, we first need to find ∇f and ∇g, where $g(x, y) = x^2 + y^2$. Then,

$$\begin{aligned} f_x &= -2xe^{-x^2-y^2}(x^2 + 2y^2) + 2xe^{-x^2-y^2} \\ &= 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) \\ f_y &= -2ye^{-x^2-y^2}(x^2 + 2y^2) + 4ye^{-x^2-y^2} \\ &= 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) \\ g_x &= 2x \\ g_y &= 2y \end{aligned}$$

Thus, our system of equations is

$$\begin{aligned} 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) &= \lambda 2x \\ 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) &= \lambda 2y \\ x^2 + y^2 &= 4 \end{aligned}$$

Solving the first equation, we can move everything to one side to get

$$\begin{aligned} 0 &= 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) - \lambda 2x \\ &= 2x(e^{-x^2-y^2}(1 - x^2 - 2y^2) - \lambda) \end{aligned}$$

By the zero-product principle, either $2x = 0$ (so $x = 0$), or $e^{-x^2-y^2}(1 - x^2 - 2y^2) - \lambda = 0$.

Similarly for the second equation, we can solve to get $y = 0$ or $e^{-x^2-y^2}(2 - x^2 - 2y^2) - \lambda = 0$.

Now, if $e^{-x^2-y^2}(1 - x^2 - 2y^2) - \lambda = 0$ and $e^{-x^2-y^2}(2 - x^2 - 2y^2) - \lambda = 0$, then we can solve both of these for λ, and set them equal to each other.

$$\begin{aligned} \lambda &= e^{-x^2-y^2}(1 - x^2 - 2y^2) \\ \lambda &= e^{-x^2-y^2}(2 - x^2 - 2y^2) \\ e^{-x^2-y^2}(1 - x^2 - 2y^2) &= e^{-x^2-y^2}(2 - x^2 - 2y^2) \\ 1 - x^2 - 2y^2 &= 2 - x^2 - 2y^2 \\ 1 &= 2 \end{aligned}$$

This indicates that we have no solutions when $x \neq 0$ or $y \neq 0$.

Notice, when $x = 0$, $y = \pm 2$ from the constraint. Similarly, when $y = 0$, $x = \pm 2$. This gives us four points to check: $(-2, 0), (2, 0), (0, -2), (0, 2)$. In the system, each point will produce a given λ value that will solve the system, so we can check these to give us our maximum and minimum values.

$$\begin{aligned} f(-2, 0) &= e^{-(-2)^2-(0)^2}((-2)^2 + 2(0)^2) &&= 4e^{-4} \\ f(2, 0) &= e^{-(2)^2-(0)^2}((2)^2 + 2(0)^2) &&= 4e^{-4} \\ f(0, -2) &= e^{-(0)^2-(-2)^2}(0)^2 + 2(-2)^2) &&= 8e^{-4} \\ f(0, 2) &= e^{-(0)^2-(2)^2}(0)^2 + 2(2)^2) &&= 8e^{-4} \end{aligned}$$

So we have a maximum of $8e^{-4}$ and a minimum of $4e^{-4}$.

13. Find the absolute extrema of $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ on $\mathscr{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$.

Solution: Our strategy will be to find the critical points of f on the interior of $\mathscr{D}$ (when $f_x = 0$ and $f_y = 0$), evaluate f at each of those critical points, then find the critical points on the boundary of $\mathscr{D}$, and evaluate f at those points.

Let's find both f_x and f_y.

$$\begin{aligned} f_x &= -2xe^{-x^2-y^2}(x^2+2y^2) + 2xe^{-x^2-y^2} \\ &= 2xe^{-x^2-y^2}(1-x^2-2y^2) \\ f_y &= -2ye^{-x^2-y^2}(x^2+2y^2) + 4ye^{-x^2-y^2} \\ &= 2ye^{-x^2-y^2}(2-x^2-2y^2) \end{aligned}$$

We will now set $f_x = 0$ to find conditions that we will use when solving $f_y = 0$.

$$\begin{aligned} 0 &= f_x \\ &= 2xe^{-x^2-y^2}(1-x^2-2y^2) \end{aligned}$$

This occurs when

$$2x = 0 \qquad \text{or} \qquad e^{-x^2-y^2} = 0 \qquad \text{or} \qquad 1-x^2-2y^2 = 0$$

Now, $e^{-x^2-y^2} = 0$ provides no solutions. If $2x = 0$, then $x = 0$. This will be our first case condition. The other case condition will occur when $1-x^2-2y^2 = 0$.

1. Suppose $x = 0$. Then

$$\begin{aligned} 0 &= f_y \\ &= 2ye^{-y^2}(2-(0)^2-2y^2 \\ &= 2ye^{-y^2}(2-2y^2) \end{aligned}$$

   Then $2y = 0$, $e^{-y^2} = 0$, or $2-2y^2 = 0$.

   If $2y = 0$, then $y = 0$. Since this is conditional upon $x = 0$, we have our first critical point at $(0,0)$.

   If $e^{-y^2} = 0$, then there are no solutions.

   If $2-2y^2 = 0$, then $y^2 = 1$, so $y = \pm 1$. This gives us two more critical points at $(0,-1)$ and $(0,1)$.

2. If $1-x^2-2y^2 = 0$, then

$$\begin{aligned} 0 &= f_y \\ &= 2ye^{-x^2-y^2}(1+1-x^2-2y^2) \\ &= 2ye^{-x^2-y^2}(1+(0)) \\ &= 2ye^{-x^2-y^2} \end{aligned}$$

   Then $2y = 0$ or $e^{-x^2-y^2} = 0$.

   If $2y = 0$, then $y = 0$. Since this is conditional upon $1-x^2-2y^2 = 0$, then $1-x^2-2(0)^2 = 0$, or $1 = x^2$. This means $x = \pm 1$. We then have two more critical points at $(-1,0)$ and $(1,0)$.

   If $e^{-x^2-y^2} = 0$, then there are no solutions.

We have five critical points of $(-1,0),(1,0),(0,0),(0,-1)$, and $(0,1)$. Evaluating each of these points, we have

$$f(0,0)=0 \qquad f(-1,0)=f(1,0)=e^{-1} \qquad f(0,-1)=f(0,1)=2e^{-1}$$

This means that the minimum of all of the points *inside* the boundary is 0, and the maximum is $2e^{-1}$.

We now move to the boundary, $x^2+y^2=4$. Then

$$\begin{aligned} e^{-x^2-y^2}(x^2+2y^2) &= e^{-(x^2+y^2)}(x^2+y^2+y^2) \\ &= e^{-4}(4+y^2) \end{aligned}$$

If we call $g(y)=e^{-4}(4+y^2)$, then we want to find the critical values of g. Note that $g'(y)=2e^{-4}y$. It follows that if $g'(y)=0$, then $y=0$. Since this is conditional upon $x^2+y^2=4$, $x^2+(0)^2=4$, so $x=\pm 2$. This gives us two critical points on the boundary: $(2,0)$, and $(-2,0)$.

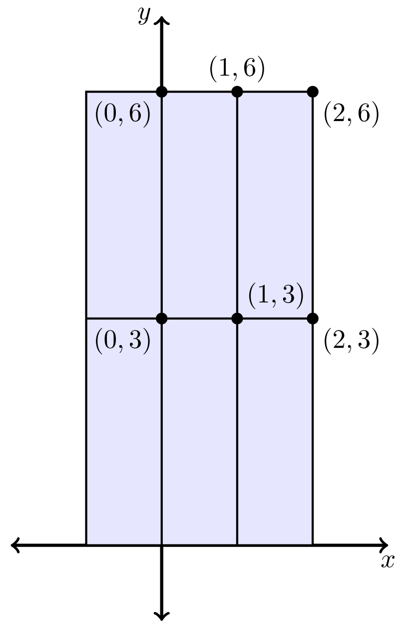
Evaluating each of these, we have

$$f(-2,0)=f(2,0)=8e^{-4}$$

Since $0<8e^{-4}<e^{-1}<2e^{-1}$, the absolute maximum of f on $\mathscr{D}$ is $2e^{-1}$, and the absolute minimum is 0.

14. Use a Riemann sum with $m = 3$ and $n = 2$ to estimate $\displaystyle\iint_R x^2 \sin(x - y)\, dA$ where $R = [-1, 2] \times [0, 6]$. Take the sample points to be upper-right corners. Round your conclusion to the nearest thousandth.

Solution: We want to subdivide R into 6 rectangles, where $\Delta x = \frac{2-(-1)}{3} = 1$ and $\Delta y = \frac{6-0}{2} = 3$.



To find our Riemann sum, we will evaluate $f(x, y) = x^2 \sin(x - y)$ at each of these six points, multiply each by $\Delta x \Delta y$, and find their sum. It follows that our Riemann sum is going to be

$$\begin{aligned}
S &= f\,(0,3)\,\Delta x\Delta y + f\,(1,3)\,\Delta x\Delta y + f\,(2,3)\,\Delta x\Delta y + f\,(0,6)\,\Delta x\Delta y + f\,(1,6)\,\Delta x\Delta y + \\
&\qquad f\,(2,6)\,\Delta x\Delta y \\
&= \Delta x\Delta y(f(0,3) + f(1,3) + f(2,3) + f(0,6) + f(1,6) + f(2,6)) \\
&= (1)(3)((0)^2\sin(0-3) + (1)^2\sin(1-3) + (2)^2\sin(2-3) + (0)^2\sin(0-6) + \\
&\qquad (1)^2\sin(1-6) + (2)^2\sin(2-6) \\
&= 3(\sin(-2) + 4\sin(-1) + \sin(-5) + 4\sin(-4)) \\
&\approx -0.867
\end{aligned}$$

15. Find the exact value of $\iint\limits_R x^2 \sin(x-y)\ dA$ where $R = [-1,2] \times [0,6]$. Then approximate this value to the nearest thousandth.

Solution: We can turn our double integral into an iterated integral as such.

$$\iint\limits_R x^2 \sin(x-y)\ dA = \int_0^6 \int_{-1}^2 x^2 \sin(x-y)\ dx\ dy$$

The order of integration given is to first integrate with respect to x, then with respect to y. This interior integral will require integration by parts a few times. Using $u = x^2$ and $dv = \sin(x-y)\ dx$, we can integrate.

$$\begin{aligned}
\int x^2 \sin(x-y)\ dx &= x^2(-\cos(x-y)) - \int 2x(-\cos(x-y))\ dx \\
&= -x^2\cos(x-y) + \int 2x\cos(x-y)\ dx \\
&= -x^2\cos(x-y) + \left(2x\sin(x-y) - \int 2\sin(x-y)\ dx\right) \\
&= -x^2\cos(x-y) + 2x\sin(x-y) + 2\cos(x-y)
\end{aligned}$$

This is the antiderivative of the integrand with respect to x. It follows that

$$\begin{aligned}
\int_0^6 \int_{-1}^2 x^2 \sin(x-y)\ dx\ dy &= \int_0^6 \left[-x^2\cos(x-y) + 2x\sin(x-y) + 2\cos(x-y)\right]_{x=-1}^{x=2}\ dy \\
&= \int_0^6 \left[(-(2)^2\cos(2-y) + 2(2)\sin(2-y) + 2\cos(2-y)) - \right. \\
&\qquad \left. (-(-1)^2\cos(-1-y) + 2(-1)\sin(-1-y) + 2\cos(-1-y))\right]\ dy \\
&= \int_0^6 (-2\cos(2-y) + 4\sin(2-y) - \cos(-1-y) - 2\sin(-1-y))\ dy \\
&= [2\sin(2-y) + 4\cos(2-y) + \sin(-1-y) + 2\cos(-1-y)]_{-1}^2 \\
&= 2\sin(-4) + 4\cos(-4) + \sin(-7) + 2\cos(-7) - 2\sin(2) - 4\cos(2) - \\
&\qquad \sin(-1) - 2\cos(-1) \\
&\approx -0.643
\end{aligned}$$

16. Let V be the volume of the solid beneath the surface $y + 2xe^y - z = 0$ and above the rectangle $R = [1,2] \times [0,2]$.

   (a) Set up an integral that represents V.

   (b) Find the exact value of V.

Solution: We may solve for z to obtain $z = y + 2xe^y$. An integral that represents V is

$$\iint_R (y + 2xe^y)\ dA$$

To find the exact value of V, we can transform this into an iterated integral and integrate.

$$\begin{aligned}
\iint_R (y + 2xe^y)\ dA &= \int_1^2 \int_0^2 (y + 2xe^y)\ dy\ dx \\
&= \int_1^2 \left[\frac{1}{2}y^2 + 2xe^y\right]_{y=0}^{y=2} dx \\
&= \int_1^2 \left[\frac{1}{2}(2)^2 + 2xe^2 - \frac{1}{2}(0)^2 - 2xe^0\right] dx \\
&= \int_1^2 (2 + 2e^2x - 2x)\ dx \\
&= \int_1^2 (2 + (2e^2 - 2)x)\ dx \\
&= \left[2x + (e^2-1)x^2\right]_1^2 \\
&= 2(2) + (e^2-1)(2)^2 - 2(1) - (e^2-1)(1)^2 \\
&= 2 + 3e^2 - 3 \\
&= 3e^2 - 1
\end{aligned}$$

17. Convert $(1, -2, 7)$ to cylindrical coordinates. Draw a three-dimensional coordinate system, and plot your point. Round your values to the nearest hundredth.

Solution: To convert rectangular coordinates to cylindrical coordinates, we use

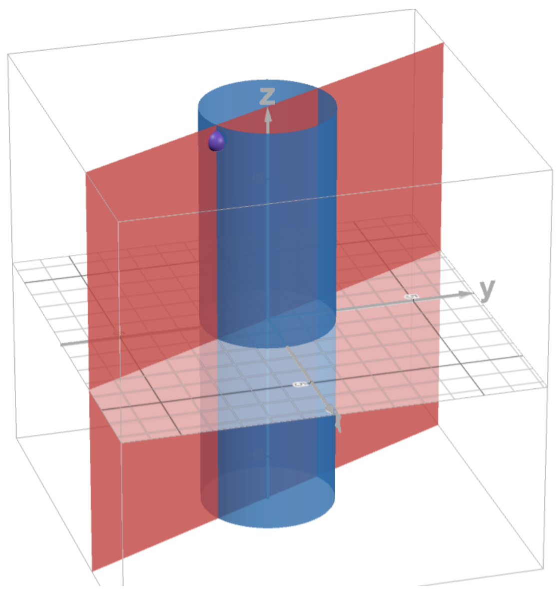
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

In this case, $r^2 = 1^2 + (-2)^2$, so $r^2 = 5$, and we choose $r = \sqrt{5}$.

For θ, we notice that $\tan\theta = \frac{-2}{1}$, so $\tan\theta = -2$. Now, $(1, -2, 7)$ is in the direction of the fourth quadrant on the xy-plane, which makes it appropriate to use the arctangent function. That is, $\theta = \arctan(-2)$.

Therefore, $(1, -2, 7)$ can be represented in cylindrical coordinates as

$$\left(\sqrt{5}, \arctan(-2), 7\right) \approx (2.236, -1.107, 5)$$



18. Convert $(-3, -1, 2)$ to spherical coordinates. Draw a three-dimensional coordinate system, and plot your point. Round your values to the nearest hundredth.

Solution: To convert rectangular coordinates to spherical coordinates, we use

$$\rho^2 = x^2 + y^2 + z^2 \qquad x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi$$

In this case, $rho^2 = (-3)^2 + (-1)^2 + (2)^2$, so $rho^2 = 14$, and so $\rho = \sqrt{14}$.
For φ, we have $z = \rho\cos\varphi$, so $2 = \sqrt{14}\cos\varphi$. Thus, $\varphi = \arccos\frac{2}{\sqrt{14}} \approx 1.007$.
For θ, we have $y = \rho \sin\varphi \sin\theta$, so

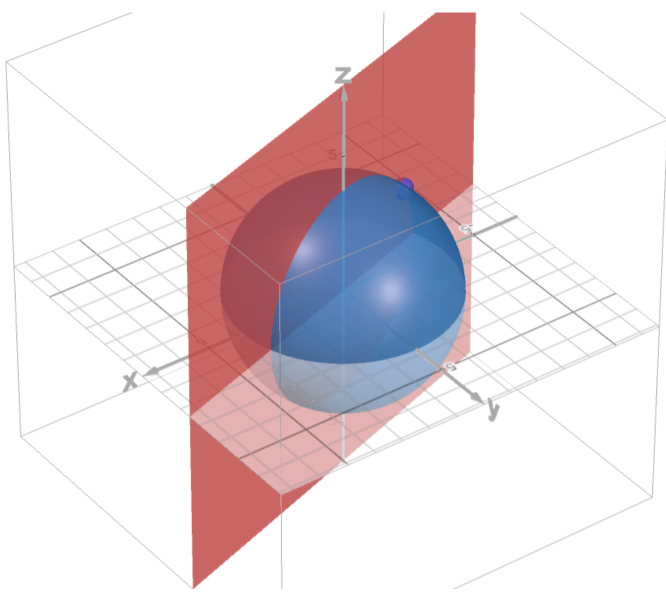
$$\begin{aligned} y &= \rho \sin\varphi \sin\theta \\ -1 &= \sqrt{14} \sin \frac{2}{\sqrt{14}} \sin\theta \\ \frac{-1}{\sqrt{14}\sin\frac{2}{\sqrt{14}}} &= \sin\theta \end{aligned}$$

Because $(-3, -1, 2)$ is in the direction of the third quadrant in the xy-plane, the arcsine function can be used to find a reference angle, but it will not produce θ itself. It follows that

$$\theta = \pi + \arcsin\left(\frac{1}{\sqrt{14}\sin\frac{2}{\sqrt{14}}}\right) \approx 3.694$$

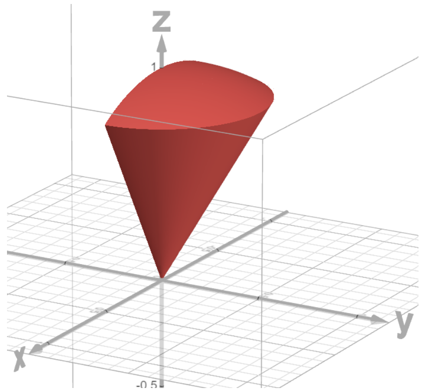
Therefore, $(-3, -1, 2)$ can be represented in spherical coordinates as

$$\left(\sqrt{14}, \pi + \arcsin\left(\frac{1}{\sqrt{14}\sin\frac{2}{\sqrt{14}}}\right), \arccos\frac{2}{\sqrt{14}}\right) \approx (3.742, 3.694, 1.007)$$



19. Sketch the solid described by $\rho \leq 1$, $0 \leq \varphi \leq \frac{\pi}{6}$, and $0 \leq \theta \leq \pi$.

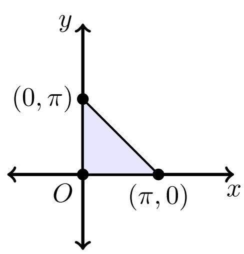
Solution: See `https://www.desmos.com/3d/6cypbtszfq`.



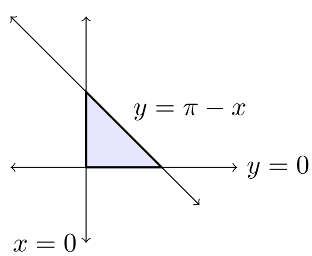
20. Evaluate $\displaystyle\iiint_E \sin y \, dV$, where E lies below the plane $z = x$ and above the triangular region whose vertices are $(0, 0, 0), (\pi, 0, 0)$ and $(0, \pi, 0)$.

Solution: In order to evaluate this integral, we must first understand E. We need inequalities for x, y, and z.

First of all, let's look at this triangular region whose vertices are $(0, 0, 0), (\pi, 0, 0)$, and $(0, \pi, 0)$. These points all lie on the xy-plane, so let's graph this triangular region.



Notice that the lines that describe this triangle are as below.



In this image, we can see that y is bounded below by $y = 0$ and above by $y = \pi - x$. Hence, $0 \leq y \leq \pi - x$. Moreover, x is bounded on the left by $x = 0$ and on the right by $x = \pi$. Thus, $0 \leq x \leq \pi$.

This leaves us with the bound for z. We know that E lies below the plane $z = x$ and about this triangular region which lives in the xy-plane. Another way to represent the xy-plane is $z = 0$. Therefore, z is bounded above by $z = x$ and below by $z = 0$, so $0 \leq z \leq x$. It follows that

$$E : \begin{cases} 0 \leq x \leq \pi \\ 0 \leq y \leq \pi - x \\ 0 \leq z \leq x \end{cases}$$

We can now transform our triple integral into an iterated integral, remembering that the integral with constant limits of integration must come last. We will then integrate (note that in our integration, we use the trigonometric identity that $\cos(\pi - x) = -\cos x$).

$$
\begin{aligned}
\iiint_E \sin y \; dV &= \int_0^{\pi} \int_0^{x} \int_0^{\pi - x} \sin y \; dy \; dz \; dx \\
&= \int_0^{\pi} \int_0^{x} [-\cos y]_{y=0}^{y=\pi-x} \; dz \; dx \\
&= \int_0^{\pi} \int_0^{x} [-\cos(\pi - x) + \cos(0)] \; dz \; dx \\
&= \int_0^{\pi} \int_0^{x} (\cos x + 1) \; dz \; dx \\
&= \int_0^{\pi} [z \cos x + z]_{z=0}^{z=x} \; dx \\
&= \int_0^{\pi} (x \cos x + x - 0 \cos x - 0) \; dx \\
&= \int_0^{\pi} (x + x \cos x) \; dx \\
&= \left[\frac{1}{2}x^2 + x \sin x + \cos x\right]_0^{\pi} \\
&= \frac{1}{2}(\pi)^2 + (\pi)\sin(\pi) + \cos(\pi) - \frac{1}{2}(0)^2 - (0)\sin(0) - \cos(0) \\
&= \frac{1}{2}\pi^2 + 0 - 1 - 0 - 0 - 1 \\
&= \frac{\pi^2}{2} - 2
\end{aligned}
$$

21. Evaluate $\iiint\limits_E \sqrt{x^2+y^2+z^2}\, dV$, where E is the solid above the cone $z = \sqrt{x^2+y^2}$ and between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$.

Solution: In order to evaluate this integral, we must first understand E.

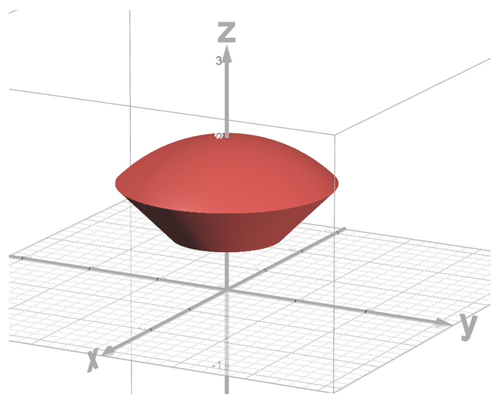
Now, we are told that our solid lies between $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$. These equations describe spheres centered at the origin of radii 1 and 2, so we can express these nicely in spherical coordinates. That is, $1 \leq \rho \leq 2$.

We are also told that E is the solid above the cone $z = \sqrt{x^2+y^2}$. This quadric surface is produced by the equation $z^2 = x^2 + y^2$, which describes a Pythagorean relationship for a $45° - 45° - 90°$ triangle. This means that $0 \leq \varphi \leq \frac{\pi}{4}$.

Lastly, our shape E is obtained by revolving entirely around the z-axis, so $0 \leq \theta \leq 2\pi$. Thus,

$$E : \begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \frac{\pi}{4} \end{cases}$$

See `https://www.desmos.com/3d/1xjkgsjotk`.



When we convert a triple integral into spherical coordinates, we must use the conversion factor. That is,

$$\begin{aligned} \iiint\limits_E \sqrt{x^2+y^2+z^2}\, dV &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_1^2 \rho\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_1^2 \rho^3 \sin\varphi \, d\rho\, d\varphi\, d\theta \end{aligned}$$

Because our limits of integration are all constant, and because our integrand can be factored into distinct functions of ρ, φ, and θ, we can split the iterated integral into three separate definite integrals.

$$\begin{aligned} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_1^2 \rho^3 \sin\varphi \, d\rho\, d\varphi\, d\theta &= \int_1^2 \rho^3 \, d\rho \int_0^{\frac{\pi}{4}} \sin\varphi\, d\varphi \int_0^{2\pi} d\theta \\ &= \left[\frac{1}{4}\rho^4\right]_1^2 [-\cos\varphi]_0^{\frac{\pi}{4}} [\theta]_0^{2\pi} \\ &= \left(\frac{1}{4}(2)^4 - \frac{1}{4}(1)^4\right)\left(-\cos\frac{\pi}{4} + \cos 0\right)(2\pi - 0) \\ &= \frac{15}{4}\left(\frac{-\sqrt{2}}{2} + 1\right)(2\pi) \\ &= \frac{15\pi(\sqrt{2}-2)}{4} \end{aligned}$$