MTH 254 Guided Notes

Damien Adams

Contents

19 Cylindrical and SphericalCoordinates **93**

20 Double Integrals Over Rectangles **97**

20.1 Single-Variable Definite Integral 97

20.2 Two-Variable Definite Integral 98

20.3 Midpoint Rule 100

20.4 Average Value 101

21 Iterated Integrals **102**

22 Double Integrals Over General Regions **105**

22.1 Generalizing the Region of Integration 105

22.2 Integrating on a General Region 106

22.3 Properties of Double Integrals 108

23 Double Integrals Over Polar Regions **109**

23.1 Polar Rectangles 109

23.2 Defining a Double Integral over a Polar Rectangle 110

23.3 Integrating over a General Polar Region 113

24 Triple Integrals **114**

24.1 Triple Integrals over a Box 114

24.2 Triple Integrals over a General Solid 115

24.3 Size and Average Values 118

25 Triple Integrals in Cylindrical and Spherical Coordinates 121

25.1 Triple Integrals in Cylindrical Coordinates 121

25.2 Triple Integrals in Spherical Coordinates 123

25.3 Rationale for the Triple Integrals Formula in Spherical Coordinates 125

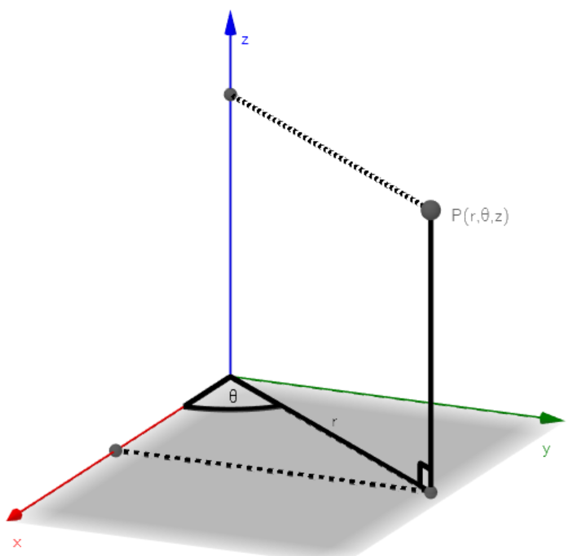
19 Cylindrical and SphericalCoordinates

Exploration: In $\mathbb{R}^2$, we typically use either Cartesian or Polar coordinates. So far, we have used an analog of Cartesian coordinates. Now, we will expand on Polar coordinates with two different systems.

Cylindrical Coordinates

We identify every points by the ordered triple (r, θ, z), where r and θ are the polar coordinates of the projection of P onto the xy-plane, and z is the directed distance from the xy-plane. That is, to navigate to P, we begin at the origin, rotate to point in the direction of P, move directly underneath or above P, and then raise or lower to the point P.

Exploration: https://www.geogebra.org/classic/X3j28ZkC



Cylindrical → Rectangular

Rectangular → Cylindrical

Example 1. Draw a set of coordinate axes for $\mathbb{R}^3$. Convert the rectangular point $P(-\sqrt{2}, -\sqrt{2}, 1)$ to cylindrical coordinates, and plot P on your axes.

Example 2. Draw a set of coordinate axes for $\mathbb{R}^3$. Convert the cylindrical point $Q\left(4, \frac{5\pi}{6}, -3\right)$ to rectangular coordinates, and plot Q on your axes.

Exercise 1. Draw a set of coordinate axes for $\mathbb{R}^3$.

a. Convert the rectangular point $P(\sqrt{3}, -1, 2)$ to cylindrical coordinates, and plot P on your axes.

b. Convert the cylindrical point $Q\left(4, \frac{2\pi}{3}, -1\right)$ to rectangular coordinates, and plot Q on your axes.

Example 3. Describe the surface whose equation is given.

a. $z = r$

b. $\theta = \dfrac{\pi}{6}$

Exercise 2. Describe the surface whose equation is given.

a. $r = 2$

b. $\theta = 1$

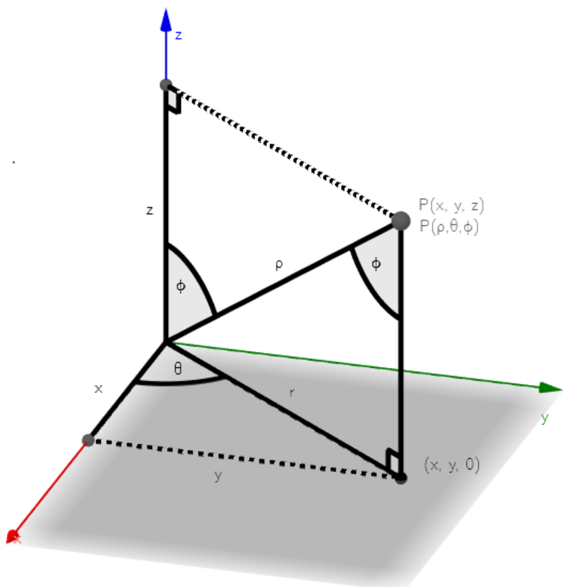
Example 4. Find an equation for the quadric surface $4x^2 + 4y^2 - z = 0$ in cylindrical coordinates. What kind of a quadric surface is this?

Exercise 3. Classify the quadric surface $3r^2 - 4z^2 = 0$ in cylindrical coordinates.

Spherical Coordinates

We identify every point by the ordered triple (ρ, θ, ϕ), where we first rotate along the xy-plane a directed angle of θ, then rotate downward from the z-axis an angle of ϕ, then travel a distance of ρ to the point P. Note that $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

Exploration: https://www.geogebra.org/classic/P2RQ7SRz



Spherical → Rectangular

Rectangular → Spherical

Example 5. Draw a set of coordinate axes for $\mathbb{R}^3$. Convert the spherical point $P\left(4, \frac{\pi}{3}, \frac{\pi}{4}\right)$ to rectangular coordinates, and plot P on your axes.

Example 6. Draw a set of coordinate axes for $\mathbb{R}^3$. Convert the rectangular point $Q(0, 2\sqrt{3}, -2)$ to spherical coordinates, and plot Q on your axes.

Exercise 4. Draw a set of coordinate axes for $\mathbb{R}^3$.

a. Convert the spherical point $P\left(1, \frac{\pi}{4}, \frac{3\pi}{4}\right)$ to rectangular coordinates, and plot P on your axes.

b. Convert the rectangular point $Q(-1, 1, -\sqrt{2})$ to spherical coordinates, and plot Q on your axes.

Example 7. Given the spherical equation $\rho = \sin\theta \sin\phi$, find a rectangular equation for the surface and identify the surface.

Exercise 5. Given the spherical equation $\rho^2 \sin^2\phi - \rho^2 \cos^2\phi = 1$, find a rectangular equation for the surface and identify the surface.

20 Double Integrals Over Rectangles

20.1 Single-Variable Definite Integral

Consider f defined on the interval $[a, b]$ and suppose for now that $f(x) \geq 0$ for all $x \in [a, b]$. Let R be the region above $[a, b]$ and underneath $y = f(x)$.

Goal: Find a formula for the area A of the region R.

Suppose f is defined on some interval $[a, b]$, and A is the area contained between the x-axis and $y = f(x)$ between $[a, b]$. Since f is defined on $[a, b]$, we can

1. Divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$.
2. Choose sample points $x_i^* \in [x_{i-1}, x_i]$ for each subinterval.
3. Form the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$ (where $f(x_i^*)\Delta x$ represents the area of each sub-rectangle).
4. $A \approx \displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$.

Definition

We define the **definite integral** of f from a to b with respect to x to be $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = A$. In the case that $f(x) \geq 0$ for all $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\,dx$ represents the area above the x-axis, below $y = f(x)$, to the right of $x = a$, and to the left of $x = b$.

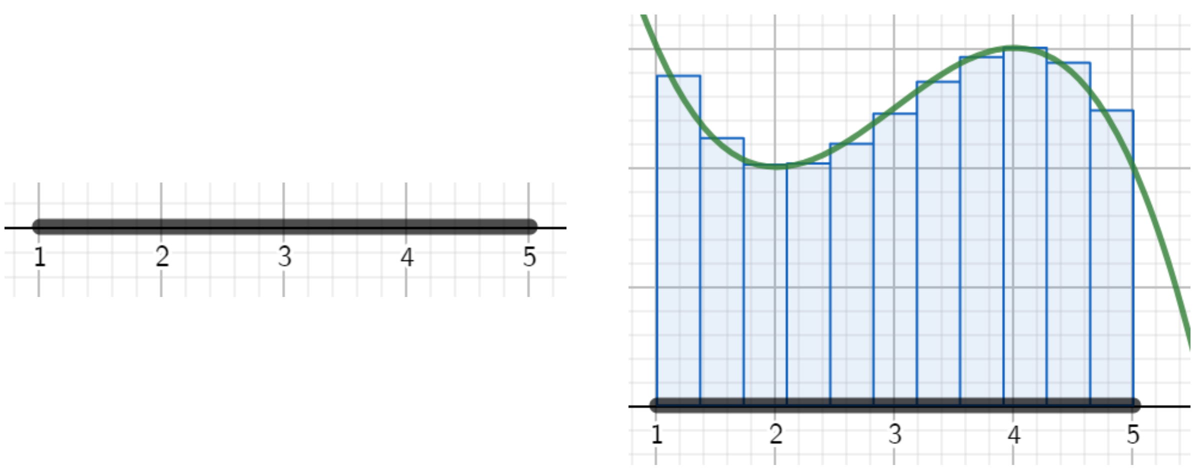


Figure 1: https://www.geogebra.org/graphing/pxhnratb

20.2 Two-Variable Definite Integral

For two-variable functions, we do not integrate on an interval but rather on a two-dimensional region. In particular, we integrate on a rectangle.

Consider f defined on a closed rectangle

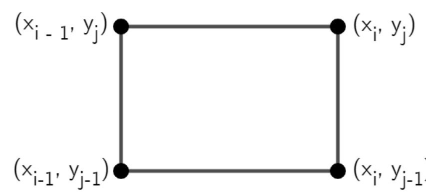
$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b], y \in [c, d]\}$$

and for now we suppose $f(x, y) \geq 0$ for all $(x, y) \in R$. Let S be the solid above R and underneath $z = f(x, y)$.

Goal: Find a formula for the volume V of S.

Suppose f is defined for all $(x, y) \in R$. Then we can

1. Subdivide R into subrectangles.
   i. Divide $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{m}$,
   ii. Divide $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = \frac{d-c}{n}$,



   iii. Define $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x \in [x_{i-1}, x_i], y \in [y_{j-1}, y_j]\}$
   iv. Each R_{ij} has area $\Delta A = \Delta x \Delta y$.
2. Choose a sample point $(x^*_{ij}, y^*_{ij}) \in R_{ij}$.
3. Form the **double Riemann sum** $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_i, y^*_j)\ \Delta A$ (where $f(x^*_i, y^*_j)\Delta A$ represents the volume of each subcolumn).
4. $\displaystyle V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_i, y^*_j)\ \Delta A.$

Definition

We define the **double integral** of f over the rectangle R to be

$$\iint_R f(x, y)\ dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_i, y^*_j)\ \Delta A$$

provided the limit exists. In the case that this limit exists, we say that f is **integrable**. Moreover, if f is integrable, and $f(x, y) \geq 0$ for all $(x, y) \in R$, then we define the **volume** of the solid S that lies above R and underneath $z = f(x, y)$ to be

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_i, y^*_j)\ \Delta A$$

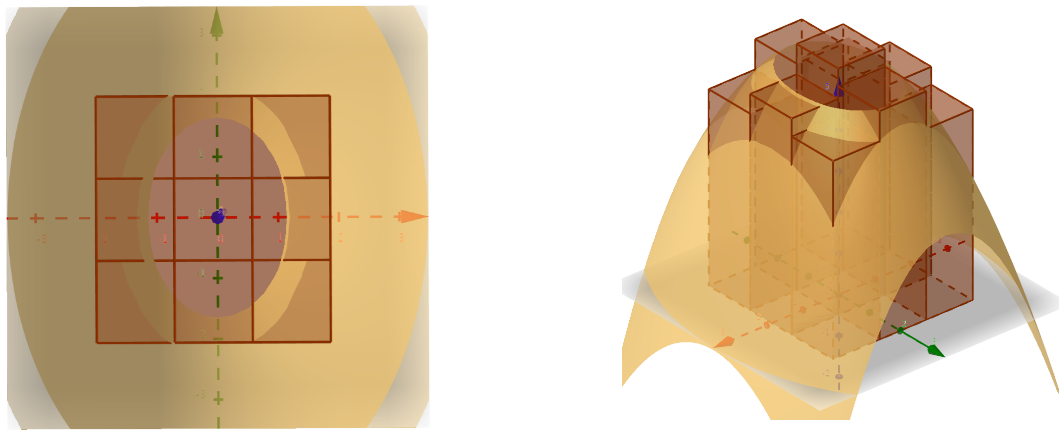


Figure 2: https://www.geogebra.org/m/kXwzQEKV

Just as in single-variable functions, the sample points (x_{ij}^*, y_{ij}^*) can be chosen to be any point in R_{ij}. If we choose (x_{ij}^*, y_{ij}^*) to be in the upper-right corner of each subrectangle, then $(x_{ij}^*, y_{ij}^*) = (x_{ij}, y_{ij})$. This simplifies the expression.

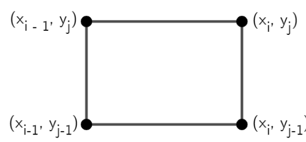


Figure 3: R_{ij}

Theorem

If f is integrable over R, then

$$\iint_R f(x,y)\ dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\ \Delta A$$

Example 1. Use a Riemann sum with $m = 2$ and $n = 3$ to estimate the volume of the solid that lies below the surface $z = \dfrac{1}{x} + \sqrt[3]{y} - 1$ and above the rectangle $R = [1, 2] \times [1, 4]$. Take the sample points to be upper-left corners. Round your conclusion to the nearest thousandth. (See: https://www.geogebra.org/3d/rdd3fmfp)

20.3 Midpoint Rule

Theorem

Midpoint Rule for Double Integrals: If f is integrable over R, then we call the double Riemann sum below the **Midpoint Rule**, and

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\ \Delta A \approx \iint_R f(x,y)\ dA$$

where $\overline{x_i}$ is the midpoint of $[x_{i-1}, x_i]$ and y_j is the midpoint of $[y_{j-1}, y_j]$.

Example 2. Use the Midpoint Rule with $m = 2$ and $n = 3$ to estimate the volume of the solid that lies below the surface $z = \dfrac{1}{x} + \sqrt[3]{y} - 1$ and above the rectangle $R = [1,2] \times [1,4]$.

Example 3. Let $R = \{(x,y) \mid 2 \leq x \leq 5, 0 \leq y \leq 2\}$. Evaluate $\displaystyle\iint_R \sqrt{4-y^2}\ dA$.

20.4 Average Value

Theorem

If f is an integrable single-variable function on $[a, b]$, then the **average value** of f on $[a, b]$ is

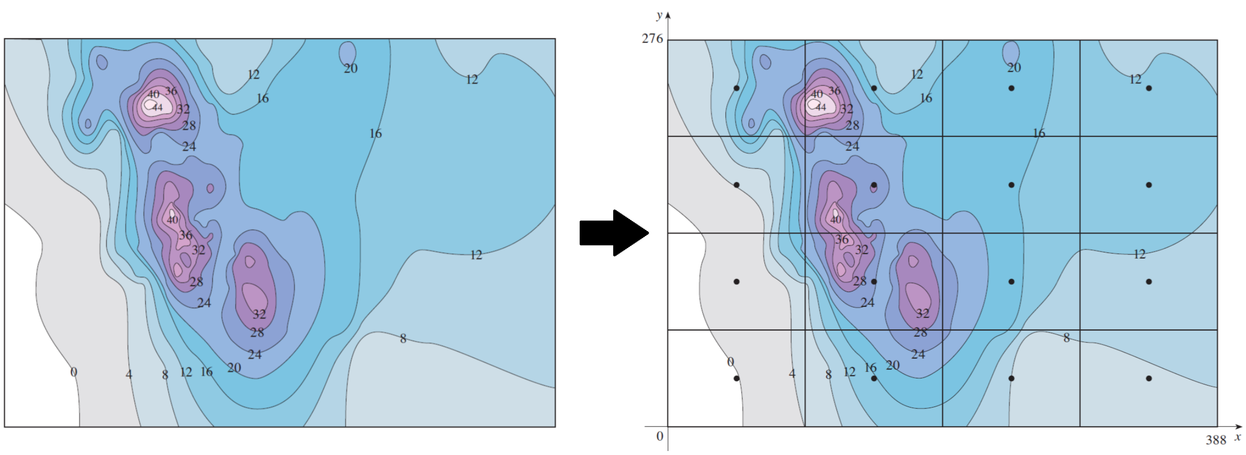
$$f_{\text{ave}} = \frac{1}{\text{size}([a,b])} \int_a^b f(x)\, dx = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Theorem

If f is an integrable two-variable function on $R = [a, b] \times [c, d]$, then the **average value** of f on R is

$$f_{\text{ave}} = \frac{1}{\text{size}(R)} \iint\limits_R f(x,y)\, dA = \frac{1}{\text{Area}(R)} \iint\limits_R f(x,y)\, dA$$

Example 4. The contour map below shows the snowfall, in inches, that fell on Colorado on December 20 – 21, 2006. The state is 388 mi east-west and 276 mi north-south. Use the contour map to estimate the average snowfall for the state on those days.



21 Iterated Integrals

Goal: Find a more efficient way to integrate $\iint\limits_R f(x,y)\,dA$.

Differentiation: Suppose f is differentiable. Then $\frac{\partial}{\partial y} f(x,y)$ is computed by fixing x and differentiating with respect to y. This procedure is called *partial differentiation with respect to y*.

Integration: Suppose f is integrable. Then $\int_a^b f(x,y)\ dy$ is computed by fixing x and integrating with respect to y. This procedure is called *partial integration with respect to y*. Since we will be evaluating an antiderivative with respect to y, there will still be x's left in the expression. That is,

$$A(x) = \int_c^d f(x,y)\ dy$$

We can now integrate with respect to x. That is

$$\int_a^b A(x)\ dx = \int_a^b \left(\int_c^d f(x,y)\ dy \right)\ dx$$

Definition

If f is an integrable function of two variables, then

$$\int_a^b \int_c^d f(x,y)\ dy\ dx = \int_a^b \left(\int_c^d f(x,y)\ dy \right)\ dx$$

is called an **iterated integral** of f with respect to y then x. Similarly,

$$\int_c^d \int_a^b f(x,y)\ dx\ dy = \int_c^d \left(\int_a^b f(x,y)\ dx \right)\ dy$$

is an iterated integral of f with respect to x then y.

Example 1. Evaluate $\int_1^2 \int_1^8 \left(\frac{1}{x} + \sqrt[3]{y} - 1 \right)\ dy\ dx$.

Example 2. Evaluate $\displaystyle\int_1^8 \int_1^2 \left(\frac{1}{x} + \sqrt[3]{y} - 1\right)\, dx\, dy$.

Theorem

Fubini's Theorem: If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Moreover, if f is bounded on R, f is discontinuous on only a finite number of smooth curves on R, and the iterated integral exists, then we can take the iterated integral in either order.

For an intuitive approach to Fubini's Theorem, see https://www.geogebra.org/m/KtskFc4a.

Example 3. Evaluate $\displaystyle\iint_R x\cos(xy)\, dA$, where $R = \left[0, \frac{\pi}{2}\right] \times [0,1]$.

Example 4. Find the volume of the solid in the first octant bounded by the cylinder $y^2 = 9 - z^2$ and the plane $x = 4$.

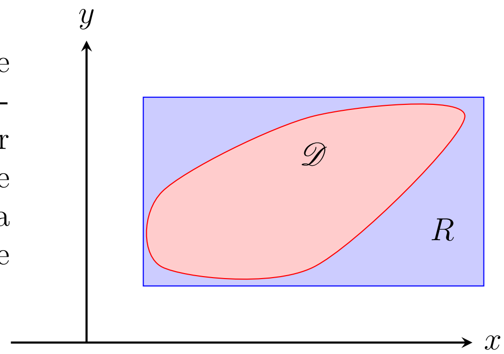
Example 5. Evaluate $\displaystyle\iint_R \frac{\sec^2 y}{x}\, dA$, where $R = [1, e] \times \left[0, \frac{\pi}{4}\right]$.

22 Double Integrals Over General Regions

Goal: Integrate f over a non-rectangular region.

22.1 Generalizing the Region of Integration

Suppose we wish to integrate f over the region $\mathscr{D}$, where $\mathscr{D}$ is a not-necessarily rectangular, bounded region. Instead of coming up with a new strategy, we can consider $\mathscr{D}$ as a region enclosed within a rectangle R. Since we wish only to integrate f over $\mathscr{D}$, we can introduce a new function F defined over R that should produce the same function as f.



$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in \mathscr{D} \\ 0 & \text{if } (x,y) \in R \backslash \mathscr{D} \end{cases}$$

Now, $\iint\limits_R f(x,y)\,dA$ is defined, and it represents the quantity that we are looking for. However, it is computationally lengthy, so we look to improve upon this.

Definition

A Cartesian region $\mathscr{D}$ is called **Type I Region (vertically simple)** if it lies between the graphs of two continuous functions of x. That is,

$$\mathscr{D} = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$.
A Cartesian region $\mathscr{D}$ is called a **Type II Region (horizontally simple)** if it lies between the graphs of two continuous functions of y. That is,

$$\mathscr{D} = \{(x,y) \in \mathbb{R}^2 \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous on $[c,d]$.

22.2 Integrating on a General Region

Theorem

If f is continuous on a Type I Region $\mathscr{D}$ such that

$$\mathscr{D} = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint\limits_{\mathscr{D}} f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

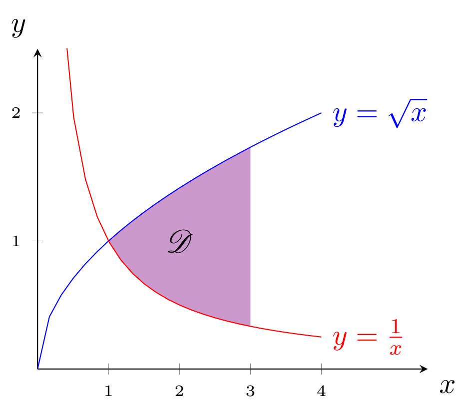
If f is continuous on a Type II Region $\mathscr{D}$ such that

$$\mathscr{D} = \{(x, y) \in \mathbb{R}^2 \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

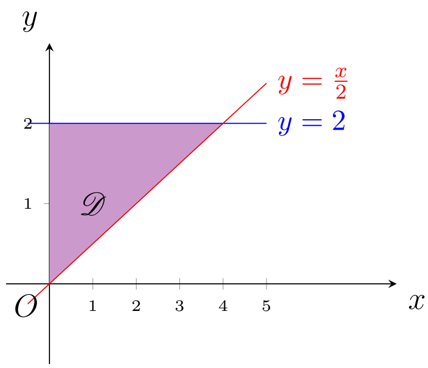
then

$$\iint\limits_{\mathscr{D}} f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example 1. Evaluate $\displaystyle\iint\limits_{\mathscr{D}} x^2 y\, dA$, where $\mathscr{D}$ is the region described below.



Example 2. Evaluate $\displaystyle\iint_{\mathscr{D}} e^{y^2}\ dA$, where $\mathscr{D}$ is the region described below.



Example 3. Sketch the domain $\mathscr{D}$ defined by $x + y \leq 12$, $y \geq 4$, and $x \geq 4$. Then evaluate $\displaystyle\iint_{\mathscr{D}} e^{x+y}\ dA$.

Example 4. Find the volume of the tetrahedron bounded by the planes $z = 2x + 3y$, $z = 0$, $x = y$, $y = \frac{x}{2}$, and $y = 2$.

22.3 Properties of Double Integrals

Theorem

If f and g are integrable function of two variables, $\mathscr{D}$ is a bounded region in $\mathbb{R}^2$, and $c \in \mathbb{R}$, then the following are true.

- $\iint\limits_{\mathscr{D}} [f(x,y) \pm g(x,y)]\ dA = \iint\limits_{\mathscr{D}} f(x,y)\ dA \pm \iint\limits_{\mathscr{D}} g(x,y)\ dA.$
- $\iint\limits_{\mathscr{D}} cf(x,y)\ dA = c \iint\limits_{\mathscr{D}} f(x,y)\ dA.$
- If $f(x,y) \geq g(x,y)$ for all $(x,y) \in \mathscr{D}$, then

$$\iint\limits_{\mathscr{D}} f(x,y)\ dA \geq \iint\limits_{\mathscr{D}} g(x,y)\ dA$$

- If $\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2$ where $\mathscr{D}_1 \cap \mathscr{D}_2 = \emptyset$, then

$$\iint\limits_{\mathscr{D}} f(x,y)\ dA = \iint\limits_{\mathscr{D}_1} f(x,y)\ dA + \iint\limits_{\mathscr{D}_2} f(x,y)\ dA$$

- $\iint\limits_{\mathscr{D}} 1\ dA = \text{Area}(\mathscr{D}).$
- If $m \leq f(x,y) \leq M$ for all $(x,y) \in \mathscr{D}$ and $A = \text{Area}(\mathscr{D})$, then

$$mA \leq \iint\limits_{\mathscr{D}} f(x,y)\ dA \leq MA$$

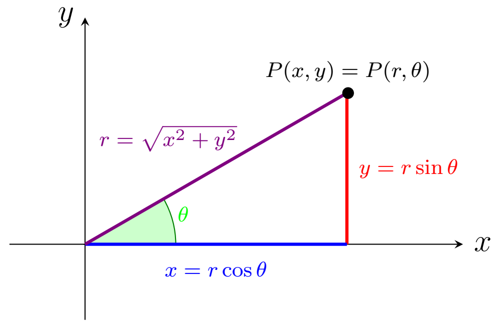
23 Double Integrals Over Polar Regions

Recall,

Polar ↔ Cartesian

If $P(x, y)$ is the Cartesian expression of the polar point $P(r, \theta)$, then

Polar → Cartesian	Cartesian → Polar
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$



23.1 Polar Rectangles

Now, a Cartesian Rectangle is defined by $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$.

Figure 4: Cartesian Rectangle

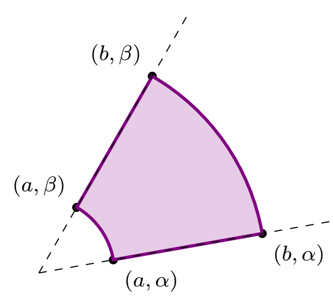


Figure 5: Polar Rectangle

Definition

A **polar rectangle** is a geometric figure of the form

$$R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

Now, the area of a sector of radius r and angle θ is $\frac{1}{2}r^2\theta$. This means that the area of a polar rectangle $R = [a, b] \times [\alpha, \beta]$ can be found by

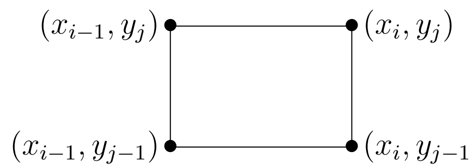
$$\begin{aligned}\text{Area}(R) &= \frac{1}{2}b^2(\beta-\alpha) - \frac{1}{2}a^2(\beta-\alpha)\\ &= \frac{1}{2}(b^2-a^2)\Delta\theta\\ &= \frac{1}{2}(b+a)(b-a)\Delta\theta\\ &= \overline{r}\ \Delta r\ \Delta\theta\end{aligned}$$

23.2 Defining a Double Integral over a Polar Rectangle

Goal: Build a double integral over a Polar rectangle.

Recall that for **integrating over a Cartesian rectangle** R, we

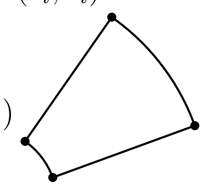
1. Subdivide R into subrectangles.
   i. Divide $[a,b]$ into m subintervals $[x_{i-1},x_i]$ of equal width $\Delta x = \frac{b-a}{m}$,
   ii. Divide $[c,d]$ into n subintervals $[y_{j-1},y_j]$ of equal width $\Delta y = \frac{d-c}{n}$,



   iii. Define $R_{ij} = [x_{i-1},x_i]\times[y_{j-1},y_j] = \{(x,y) \mid x\in[x_{i-1},x_i], y\in[y_{j-1},y_j]\}$
   iv. Each R_{ij} has area $\Delta A = \Delta x\Delta y$.
2. Choose a sample point $(x_{ij}^*, y_{ij}^*)\in R_{ij}$.
3. Form the **double Riemann sum** $\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\ \Delta A$ (where $f(x_{ij}^*, y_{ij}^*)\Delta A$ represents the volume of each subcolumn).
4. $V \approx \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\ \Delta A$.
5. Define $\displaystyle\iint_R f(x,y)\ dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\ \Delta A$.

We now work through the *same process* over a polar rectangle R. If f is an integrable funciton of x and y, then we

1. Subdivide R into (polar) subrectangles.



   i. Divide $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = \frac{b-a}{m}$,
   ii. Divide $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = \frac{\beta-\alpha}{n}$,
   iii. Define $R_{ij} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j]$
   iv. Each R_{ij} has area $\Delta A = \overline{r_i}\ \Delta r\ \Delta\theta$.

2. Choose sample points with $r^*_{ij} = \overline{r_i}$ and $\theta^*_{ij} = \overline{\theta_j}$, so $(r^*_{ij}, \theta^*_{ij}) = (\overline{r_i}, \overline{\theta_j})$ are the centers of R_{ij}.

3. Form the double Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r^*_{ij}\cos\theta^*_{ij}, r^*_{ij}\sin\theta^*_{ij})\ \Delta A$$

   (where $f(r^*_{ij}\cos\theta^*_{ij}, r^*_{ij}\sin\theta^*_{ij})\ \Delta A$ represents the volume of each "subcolumn").

4. $V \approx \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{r_i}\cos\overline{\theta_j}, \overline{r_i}\sin\overline{\theta_j})\ \Delta A = \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{r_i}\cos\overline{\theta_j}, \overline{r_i}\sin\overline{\theta_j})\ \overline{r_i}\ \Delta r\ \Delta\theta.$

5. Therefore,

$$\begin{aligned}\iint_R f(x,y)\ dA &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{r_i}\cos\overline{\theta_j}, \overline{r_i}\sin\overline{\theta_j})\ \overline{r_i}\ \Delta r\ \Delta\theta \\ &= \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta) r\ dr\ d\theta\end{aligned}$$

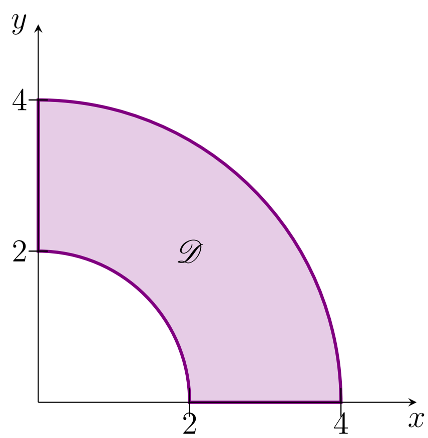
Theorem

If f is a continuous function of x and y on a polar rectangle $R = [a, b] \times [\alpha, \beta]$, with $\beta - \alpha \in [0, 2\pi]$, then

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta) r\ dr\ d\theta$$

where $x = r\cos\theta$ and $y = r\sin\theta$.

Example 1. Find the volume of the solid that lies above the quarter-annulus depicted below and beneath the plane $z = x + y$.



Example 2. Find the volume of the region bounded by the xy-plane and the paraboloid $z = 8 - 2x^2 - 2y^2$.

23.3 Integrating over a General Polar Region

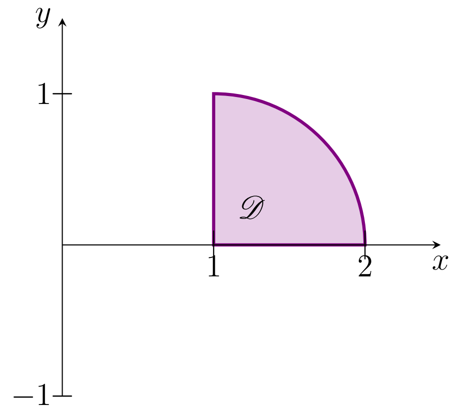
Theorem

If f is continuous on a polar region of the form

$$\mathscr{D} = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

then $\displaystyle\iint_{\mathscr{D}} f(x,y)\ dA = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\ dr\ d\theta.$

Example 3. Evaluate $\displaystyle\iint_{\mathscr{D}} \frac{8}{(x^2+y^2)^2}\ dA$ where $\mathscr{D}$ is depicted below.



24 Triple Integrals

24.1 Triple Integrals over a Box

Previously,

- $\displaystyle\int_a^b f(x)\,dx$ integrates f over an interval – a subset of $\mathbb{R}$.
- $\displaystyle\iint_{\mathscr{D}} f(x,y)\,dA$ integrate f over a region – a subset of $\mathbb{R}^2$.

Via analogy (rather than exploration), we make the following definition.

Definition

The **triple integral of f over the box** B is

$$\iiint_{\mathcal{B}} f(x,y,z)\,dV = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$$

provided the limit exists.

Just as in double integrals, Fubini's Theorem applies to triple integrals.

Theorem

Fubini's Theorem: If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_{\mathcal{B}} f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note that the limits of integration are constants for x, y, and z.

Example 1. Evaluate the integral $\displaystyle\iiint_{\mathcal{B}} (xy + z^2)\,dV$, where $\mathcal{B} = \{(x,y,z) \mid 0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3\}$.

24.2 Triple Integrals over a General Solid

Just as in double integration, limits of integration do not necessarily need to be constant, and we use the same procedure as before to deal with such situations. The strategy to deal with such situations is dependent upon the solid of which we are integrating over. As long as the region E that we are integrating over is bounded, we can enclose E within a larger box B and define a new function

$$F(x,y,z) = \begin{cases} f(x,y,z) & ,(x,y,z) \in E \\ 0 & ,(x,y,z) \notin E \end{cases}$$

It follows that $\iiint\limits_{\mathcal{W}} f(x,y,z)\ dV = \iiint\limits_{\mathcal{B}} F(x,y,z)\ dV$. This integral will exist as long as the boundary on $\mathcal{W}$ is *reasonably smooth*, and this integral will have nearly all of the same properties as double integrals.

Definition

We define the **triple integral of f over a general solid** $\mathcal{W}$ to be $\iiint\limits_{\mathcal{W}} f(x,y,z)\ dV = \iiint\limits_{\mathcal{B}} F(x,y,z)\ dV$ where

$$F(x,y,z) = \begin{cases} f(x,y,z) & ,(x,y,z) \in \mathcal{W} \\ 0 & ,(x,y,z) \notin \mathcal{W} \end{cases}$$

provided the integral exists.

In order to evaluate a triple integral over a general solid, we observe what sort of a region we have. For subsets of $\mathbb{R}^2$, we have Type I and Type II regions. For subsets of $\mathbb{R}^3$, we have Type I, Type II, and Type III regions.

Definition

Let $\mathcal{W}$ be a solid region in $\mathbb{R}^3$, and let $\mathscr{D}$ be the projection of E onto the xy-plane.

- The solid $\mathcal{W}$ is **Type I solid region** if it lies between the graphs of two continuous functions of x and y. That is,

$$\mathcal{W} = \{(x, y, z) \mid (x, y) \in \mathscr{D}, u_1(x, y) \leq z \leq u_2(x, y)\}$$

- The solid $\mathcal{W}$ is **Type II solid region** if it lies between the graphs of two continuous functions of y and z. That is,

$$\mathcal{W} = \{(x, y, z) \mid (y, z) \in \mathscr{D}, u_1(y, z) \leq x \leq u_2(y, z)\}$$

- The solid $\mathcal{W}$ is **Type III solid region** if it lies between the graphs of two continuous functions of x and z. That is,

$$\mathcal{W} = \{(x, y, z) \mid (x, z) \in \mathscr{D}, u_1(x, z) \leq y \leq u_2(x, z)\}$$

The strategy for evaluating a double integral is to turn it into two single-variable integrals.

Strategy

The strategy for evaluating a triple integral is to turn it into a single-variable integral and a double integral. For example, if $\mathcal{W}$ is a Type I solid region, then

$$\iiint\limits_{\mathcal{W}} f(x, y, z)\ dV = \iint\limits_{\mathscr{D}} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz \right)\ dA$$

Upon integration, the inside integral will leave a function of x and y, leading to a double integral that we know how to handle.

Example 2. Evaluate $\iiint\limits_{\mathcal{W}} z\ dV$, where $\mathcal{W}$ is the region between the planes $z = x + y$ and $z = 3x + 5y$ lying over the rectangle $\mathscr{D} = [0, 3] \times [0, 2]$.

Example 3. Evaluate $\iiint\limits_{\mathcal{W}} z\ dV$, where $\mathcal{W}$ is the region between the planes $z = x + y$ and $z = 3x + 5y$ lying over the triangle on the xy-plane whose vertices are $(0, 0)$, $(1, 0)$, and $(0, 1)$.

24.3 Size and Average Values

As mentioned before, triple integrals inherit a lot of the same properties as double and single integrals. In particular, we have notions of size and average value.

Theorem

Integrating 1 will produce the size of the region of integration. That is,

- If integrating over an interval in $\mathbb{R}$, then

$$\text{size}([a,b]) = \int_a^b 1 \, dx$$

where the size is the length of the interval.

- If integrating over a planar region in $\mathbb{R}^2$, then

$$\text{size}(\mathscr{D}) = \iint_{\mathscr{D}} 1 \, dA$$

where the size is the area of the region.

- If integrating over a solid $\mathcal{W}$ in $\mathbb{R}^3$, then

$$\text{size}(\mathcal{W}) = \iiint_{\mathcal{W}} 1 \, dV$$

where the size is the volume of the solid.

Moreover, the average value of an integrable function f can be found by integrating over the region of integration and dividing by the size of that region. That is,

- If f is an integrable function of x, then

$$\overline{f}_{[a,b]} = \frac{1}{\text{size}\,[a,b]} \int_a^b f(x) \, dx$$

- If f is an integrable function of x and y, then

$$\overline{f}_{\mathscr{D}} = \frac{1}{\text{size}\,\mathscr{D}} \iint_{\mathscr{D}} f(x,y) \, dA$$

- If f is an integrable function of x, y, and z, then

$$\overline{f}_{\mathcal{W}} = \frac{1}{\text{size}\,\mathcal{W}} \iiint_{\mathcal{W}} f(x,y,z) \, dV$$

Example 4. A crystal $\mathcal{W}$ is pulled extracted out of a cold case and set on its side. A coordinate system is placed around the crystal so that it exists in the first octant bounded by the coordinate planes, $x + z = 1$, and $x + y + z = 3$. The temperature at every point in the crystal is given by $T(x, y, z) = x$, measured in $^\circ$ C.

a. Find the volume of the crystal.

b. Find the average temperature of the crystal.

Example 5. Find the value of the integral of $f(x, y, z) = x$ over the region $\mathcal{W}$ bounded above by $z = 4 - x^2 - y^2$ and below by $z = x^2 + 3y^2$ in the first octant.

25 Triple Integrals in Cylindrical and Spherical Coordinates

25.1 Triple Integrals in Cylindrical Coordinates

Recall Cylindrical Coordinates: https://www.geogebra.org/3d/yesa2uja

Cylindrical → Rectangular	**Rectangular → Cylindrical**
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = z$	$z = z$

Theorem

Suppose $\mathcal{W}$ is a Type I solid region whose projection $\mathscr{D}$ onto the xy-plane has polar description

$$\mathscr{D} = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

Then

$$\begin{aligned}\iiint_{\mathcal{W}} f(x,y,z)\, dV &= \iint_{\mathscr{D}} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right) dA \\ &= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\; dz\; dr\; d\theta \\ &= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{z_1(r,theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\; dz\; dr\; d\theta\end{aligned}$$

where $h_i(x,y) = u_i(r\cos\theta, r\sin\theta) = z_i(r,\theta)$ are different expressions for the same relationship.

Example 1. Evaluate $\displaystyle\iiint_{\mathcal{W}} z\sqrt{x^2+y^2}\ dV$, where W is the cylinder $r^2 \leq 4$ with $1 \leq z \leq 5$.

Example 2. Evaluate $\displaystyle\iiint_{\mathcal{W}} z\ dV$, where $\mathcal{W}$ is the region within the cylinder $x^2 + y^2 \leq 4$ with $0 \leq z \leq y$.

25.2 Triple Integrals in Spherical Coordinates

Recall Spherical Coordinates: https://www.geogebra.org/3d/qjk6hmgz

Spherical → Rectangular	Rectangular → Spherical
$x = \rho \sin\varphi \cos\theta$ $y = \rho \sin\varphi \sin\theta$ $z = \rho \cos\varphi$	$\rho^2 = x^2 + y^2 + z^2$ $\tan\theta = \dfrac{y}{x}$ $\varphi = \arccos\dfrac{z}{\rho}$

Whereas with double integration in polar coordinates we integrated over a polar rectangle, here we triple integrate in spherical coordinates over a spherical wedge.

Definition

A **spherical wedge** is a solid described in spherical coordinates by $\mathcal{W} = \underbrace{[a,b]}_{\rho} \times \underbrace{[\alpha,\beta]}_{\theta} \times \underbrace{[\chi,\psi]}_{\varphi}$, where $a \geq 0$, $0 \leq \beta - \alpha \leq 2\pi$, and $0 \leq \psi - \chi \geq \pi$.

If we generalize a spherical wedge to allow ρ to vary between to surfaces dependent upon θ and φ, then we get the following theorem.

Theorem

Suppose f is integrable over a region $\mathcal{W}$ defined by

$$\mathcal{W} = \{(\rho,\theta,\varphi) \mid \alpha \leq \theta \leq \beta, \chi \leq \varphi \leq \psi, \rho_1(\theta,\varphi) \leq \rho \leq \rho_2(\theta,\varphi)\}$$

then

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV = \int_{\chi}^{\psi}\int_{\alpha}^{\beta}\int_{\rho_1(\theta,\varphi)}^{\rho_2(\theta,\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi d\rho\, d\theta\, d\varphi$$

A rationale is provided on the last page of this section.

Example 3. Compute the integral of $f(x, y, z) = x^2 + y^2$ over the sphere $\mathcal{S}$ of radius 3 centered at the origin.

Example 4. An ice cream filling a cone and making a spherical top can be approximately modeled according to the surfaces $z^2 = x^2 + y^2$ and $z = x^2 + y^2 + z^2$. It is known that the ice cream top is represented by a hemisphere, and two opposite edges make a right angle at the tip of the cone. Find the volume of the ice cream. (See https://www.geogebra.org/3d/vhqabgzk)

25.3 Rationale for the Triple Integrals Formula in Spherical Coordinates

Define a spherical wedge $\mathcal{W} = [a, b] \times [\alpha, \beta] \times [\chi, \psi]$, where $a \geq 0$, $0 \leq \beta - \alpha \leq 2\pi$, and $0 \leq \psi - \chi \geq \pi$. Divide $\mathcal{W}$ into subwedges $\mathcal{W}_{ijk}$ by cutting $\mathcal{W}$ with equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\varphi = \varphi_k$. Then $\mathcal{W}_{ijk}$ can be approximated by a rectangular box of dimensions $A \times B \times C$.

Now, $A = \Delta\rho$ (side extending away from the origin). For the side rotating according to θ, B, we have the length of an arc of a circle. The radius of that circle is r, and the angle of that arc is $\Delta\theta$. Converting r to spherical coordinates, we get that $r = \rho_i \sin\varphi_k$. Since the length of the arc of a circle is the product of the radius with the angle subtended by it, $B = \rho_i \sin\varphi_k \Delta\theta$. Lastly, we have the side rotating according to φ, C. For this side, we also have an arc of a circle whose radius is ρ_i, and the angle subtended by it is $\Delta\varphi$. It follows that $C = \rho_i \Delta\varphi$. Thus, an approximation for the volume of $\mathcal{W}_{ijk}$ is

$$\begin{aligned}\text{size}(\mathcal{W}) &\approx \Delta\rho \times \rho_i \Delta\varphi \times \rho_i \sin\varphi_k \Delta\theta \\ &= \rho_i^2 \sin\varphi_k \Delta\rho \ \Delta \ \Delta\theta \ \Delta\varphi\end{aligned}$$

We now consider the Riemann sum for $f(x, y, z)$. That is

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \ \Delta V$$

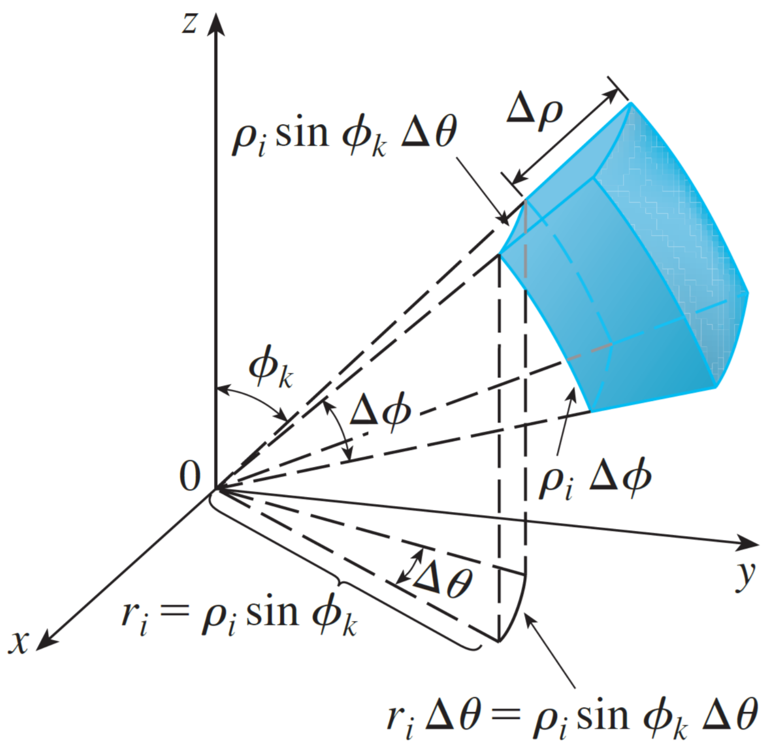
$$= \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho_i \sin\varphi_k \cos\theta_j, \rho_i \sin\varphi_k \sin\theta_j, \rho_i \cos\varphi_k)\rho_i^2 \sin\varphi_k \ \Delta\rho \ \Delta\theta \ \Delta\varphi$$

Taking limits as $\ell, m, n \to \infty$, we get

$$\iiint\limits_{\mathcal{W}} f(x, y, z) \ dV$$

$$= \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho_i \sin\varphi_k \cos\theta_j, \rho_i \sin\varphi_k \sin\theta_j, \rho_i \cos\varphi_k)\rho_i^2 \sin\varphi_k \ \Delta\rho \ \Delta\theta \ \Delta\varphi$$

$$= \int_{\chi}^{\psi} \int_{\alpha}^{\beta} \int_{\rho_1(\theta,\varphi)}^{\rho_2(\theta,\varphi)} f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)\rho^2 \sin\varphi d\rho \ d\theta \ d\varphi$$



Spherical Wedge – Stewart, *Concepts & Contexts*, 4E, Figure 7, pp. 885