MTH 254 Guided Notes

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Contents

12 Multivariable Functions

12.1 Domains and Graphs of Multivariable Functions

Definition

A real-valued **function in two variables**, f, is a function whose domain is a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$.

Example 1. Determine and sketch the domain of $f(x, y) = \sqrt{2x^2 + 2y^2 - 8}$. Then graph f in GeoGebra.

12.2 Level Curves & Contour Maps

Definition

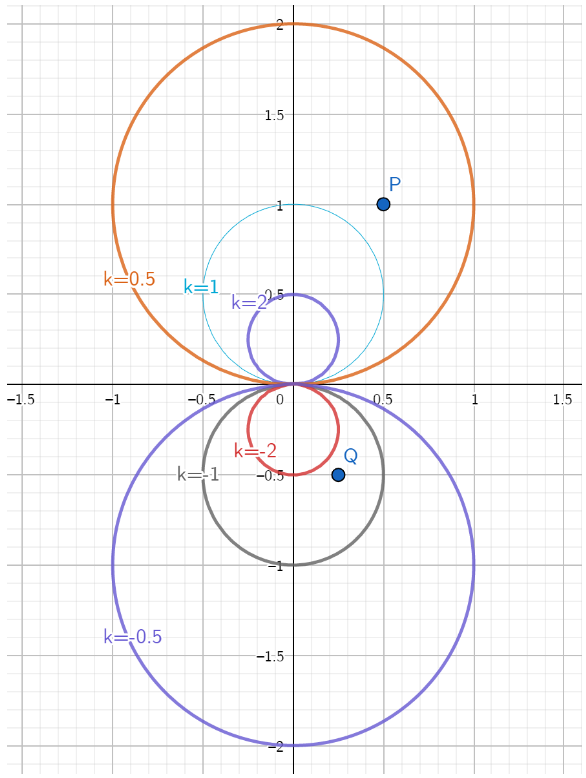
A **level curve** of a function f of two variables is a curve whose equation is $f(x, y) = k$, where $k \in \mathbb{R}$.

Level curves are useful for visualizing a function's graph in a lower-dimensional form.

Definition

A **contour map** of a real-valued function f in two variables is a graph of several level curves of f.

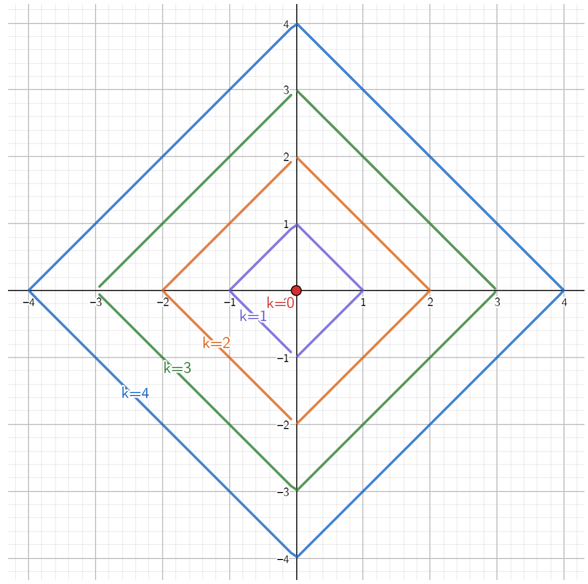
Example 2. A contour map for a function f is given below. Use it to approximate the values of $f(0.5, 1)$ and $f(0.25, -0.5)$.



Example 3. Let $f(x, y) = 1 + 2x^2 - 3y$. Draw a contour map for f for $k = -4, -2, 0, 2, 4$. Label each curve with the appropriate k-value.

12.3 Curve-Sketching from Level Curves

Example 4. A contour map of $f(x, y)$ is shown below. Use it to sketch a graph of $z = f(x, y)$.



12.4 Function of Three Variables

Definition

A real-valued **function in three variables**, f, is a function whose domain is a subset of $\mathbb{R}^3$ and whose range is a subset of $\mathbb{R}$.

The traditional graph of a function in n variables requires an $(n+1)$-dimensional coordinate space. This makes it exceedingly difficult as dimensions increase. We will use a generalization of level curves to visualize a function in three-variables.

Definition

A **level surface** of a function f of three variables is a surface whose equation is $f(x, y, z) = k$, where $k \in \mathbb{R}$.

Example 5. Let $f(x, y, z) = \ln(36 - x^2 - y^2 - z^2)$. Evaluate $f(1, -2, 3)$, and determine the domain of f.

Example 6. Describe the level surfaces of $f(x, y, z) = -x^2 + 2y^2 - 3z^2$.

12.5 Plotting Level Curves in GeoGebra

To plot level curves in GeoGebra, go to the GeoGebra graphing calculator at www.geogebra.org/graphing.

Create a slider for k:

```
k = slider( MIN , MAX , STEP )
```

where `MIN` and `MAX` are the smallest and largest values that k will scroll between. The `STEP` part indicates how much k will change as it increases from `MIN` to `MAX`.

After creating the slider, create your function:

```
f(x, y) = [ FUNCTION ]
```

where `[ FUNCTION ]` is your given function. Make sure to hit `Enter`, or else it may not work properly.

Create the graph $k = f(x, y)$ by typing that in:

```
k = f(x, y)
```

In order to view different level curves, slide k to take on several values. If you want to see a bunch of them simultaneously, click on the graph, then click on ⋮ , and choose "Show trace". When k slides, the resulting graphs will remain.

If you want to clear the traces, right click on the coordinate system, and choose "Clear all Traces".

13 Limits and Continuity

To find a limit in one variable, say as $x \to a$, we check to see if the function produces the same output as x approaches a from *both* sides.

To find a limit in two variables, say as $(x, y) \to (a, b)$, we check to see if the function produces the same output as (x, y) approaches (a, b) from *all* directions.

Definition

Given a function in two variables, f, we say that the **limit as** (x, y) **approaches (the point)** (a, b) **is** L if we can make the values of $f(x, y)$ as close to L as we like by taking (x, y) to be sufficiently close to (a, b) (but not necessarily equal to (a, b)). We write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

13.1 Pathways

Example 1. Investigate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-2xy^2}{x^2 + y^4}$ by finding path-specific limits. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-2xy^2}{x^2 + y^4}$ does not exist.

Proving that a limit exists means that we need to show that *regardless of the path that is chosen*, the limit will always be the same. This can be done in several ways, but here is one strategy – the Squeeze Theorem.

Example 2. Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{-2xy^2}{x^2+y^2}$.

13.2 Continuity

Definition

A function of two variables, f, is **continuous at (the point)** (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

That is, f is continuous at (a,b) if taking the limit is the same as evaluation.
We say that f is **continuous on (the set)** D if f is continuous at every point in D.

Theorem

Suppose f, g are both continuous on their domain. The following functions of two variables are continuous on their domain:

- $f \pm g$
- fg
- $\dfrac{f}{g}$
- $f \circ g$
- cf, where $c \in \mathbb{R}$
- Polynomials
- Rational Functions
- Radical Functions
- Exponential Functions
- Logarithmic Functions
- Trigonometric Functions
- Inverse Trigonometric Functions

Example 3. Evaluate $\displaystyle\lim_{(x,y)\to(3,5)} \frac{x^2 \sin\left(\frac{\pi}{2}y\right) - 3e^x \ln y}{\sqrt{x^2+xy+y^2}-5}$.

Example 4. Determine the set of points for which f is continuous.

$$f(x,y) = \begin{cases} \dfrac{-2xy^2}{x^2+y^4} & ,(x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

14 Partial Derivatives

14.1 Definitions

Given a curve, we can typically find a tangent *line* which is determined by its slope. Given a surface, we can typically find a tangent *plane*, which is determined by two vectors and thus two slopes. We typically resort to using a vector in the x direction and one in the y direction.

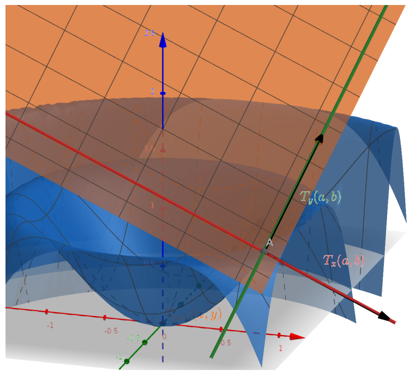


Figure 1: https://www.geogebra.org/3d/bvv8jqjp

Definition

Given a function of two variables, f, we define the **partial derivative of f with respect to x at** (a, b), $f_x(a, b)$, to be

$$\begin{aligned} f_x(a,b) &= \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x - a} \\ &= \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} \end{aligned}$$

This quantity represents the slope of the surface determined by f in the x-direction. Similarly, the **partial derivative of f with respect to y at** (a, b), $f_y(a, b)$, to be

$$\begin{aligned} f_y(a,b) &= \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y - b} \\ &= \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h} \end{aligned}$$

Moreover, we can expand these definitions to functions. The **partial derivatives of f, f_x and f_y** are defined by

$$\begin{aligned} f_x &= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ f_y &= \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \end{aligned}$$

14.2 Notation & Strategies

Notation

We notate partial derivatives in several ways. If $z = f(x, y)$, here are some notations.

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Strategy

If $z = f(x, y)$, then the way we find f_x or f_y is to regard the other variable(s) as a constant and differentiate with respect to the indicated variable. Follow *all* single-variable derivative rules.

Example 1. Let $f(x, y) = 3x^3 - x^2y - y^3$. Find

a. f_x

b. f_y

Example 2. Let $f(x, y) = \arctan \frac{y}{x}$. Find the following partial derivatives and interpret them as slopes.

a. $f_x(2, 3)$

b. $f_y(2, 3)$

Example 3. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ from the implicit equation $\sin^2 x + \cos^2 y + e^z \ln z = 1$.

Note: For functions with more variables, we follow the same principles.

Example 4. Find $f_z(x, y, z, t)$ if $f(x, y, z, t) = 2(\tan x)(\arctan y)(e^{-z})(\ln t)$.

14.3 Higher-Order Partial Derivatives

Definition

For **higher-order partial derivatives**, we use the same principle. That is

$$\begin{aligned}
f_{xx} = (f_x)_x &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2 z}{\partial x^2} \\
f_{xy} = (f_x)_y &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial^2 z}{\partial y \partial x} \\
f_{yx} = (f_y)_x &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 z}{\partial x \partial y} \\
f_{yy} = (f_y)_y &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} &= \frac{\partial^2 z}{\partial y^2} \\
f_{xyz} = (f_{xy})_z &= \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial z \partial y \partial x}
\end{aligned}$$

Other similar formulas follow.

Example 5. Let $f(x,y) = 3x^3 - x^2y - y^3$. Find

a. f_{xx} b. f_{xy} c. f_{yx} d. f_{yy}

Theorem

Clairaut's Theorem: If f is defined on a disk D that contains the point (a,b) and f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

Example 6. Let $f(x,y) = \cos(3x+5y)$. Find $f_{xyxyy}(0,0)$.

15 Tangent Planes

15.1 Tangent Planes

Recall that an equation of a plane with normal vector $\mathbf{n} = \langle a, b, c\rangle$ at the point (x_0, y_0, z_0) is

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0 \qquad \Longleftrightarrow \qquad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Definition

Suppose a surface S has equation $z = f(x, y)$ with point $P(x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$ and f has continuous partial derivatives. The **tangent plane** to S at P is the plane whose equation is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Moreover, two alternate forms of this equation are

$$\begin{aligned} 0 &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + (-1)(z - z_0) \\ z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \end{aligned}$$

Note that $\langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle$ is a normal vector to the tangent plane.

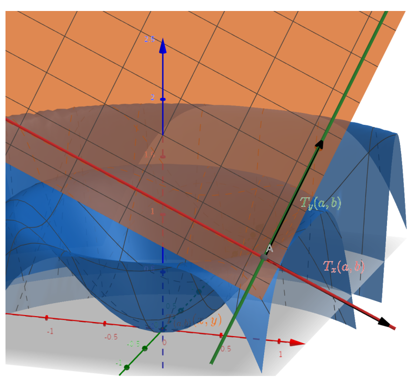


Figure 2: https://www.geogebra.org/3d/bvv8jqjp

Example 1. Let $f(x, y) = 3x^3 - x^2y - y^3$. Find the tangent plane to the surface $z = f(x, y)$ at $(1, 2, -7)$.

Example 2. Let $\sin^2 x + \cos^2 y + e^z \ln z = 1$. Find the tangent plane at the point $(1, 1, 1)$.

15.2 Linear Approximations & Differentials

In the past, you've learned of Taylor polynomials as approximations for a function at a point. Any first-degree Taylor polynomial to $f(x)$ at $x = a$ is known as a *linearization* of f at a, and its graph is the tangent line to f at a. That is, if f is differentiable at a, then

$$L(x) = f(a) + f'(a)(x - a)$$

is the linearization of f at a.

For a function of 2 variables, a linearization of a surface is not a line but a plane.

Definition

Suppose a surface S has equation $z = f(x, y)$ with point $P(x_0, y_0, z_0)$, where f has continuous partial derivatives. The **linearization** of f at (x_0, y_0) is the function whose graph is the tangent plane to f at (x_0, y_0). That is, the linearization of f at (x_0, y_0) is

$$L_{(x_0,y_0)}(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Moreover, the approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is the **linear approximation** (or **tangent plane approximation**) of f at (x_0, y_0). That is, for points near (x_0, y_0), $f(x, y) \approx L_{(x_0,y_0)}(x, y)(x, y)$.

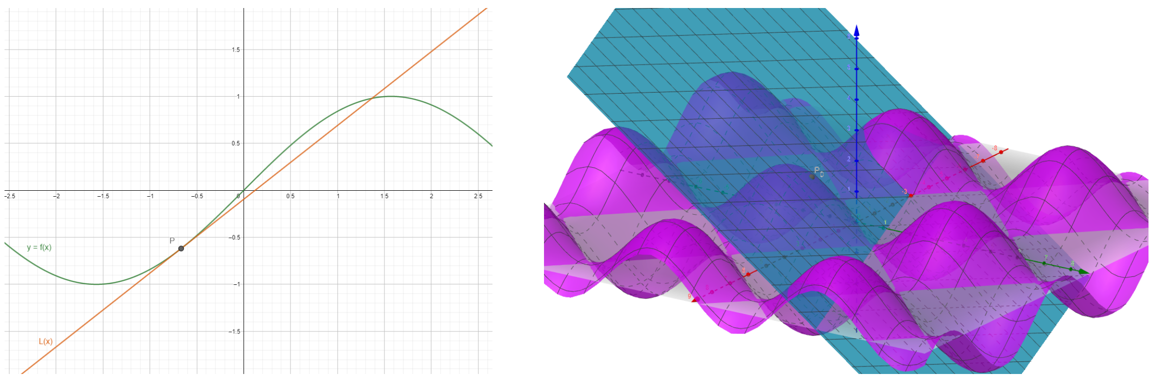


Figure 3: https://www.geogebra.org/graphing/bsbjn5cd & https://www.geogebra.org/3d/bnpb9bkk

As (x, y) changes from (x_0, y_0) to (x_1, y_1), then f changes correspondingly. We say that

$$\Delta z = f(x_1, y_1) - f(x_0, y_0) = z_1 - z_0$$

This value is often difficult to compute. However, we can using linearizations to approximate Δz. Correspondingly, we say that

$$dz = L_{(x_0,y_0)}(x_1, y_1) - L_{(x_0,y_0)}(x_0, y_0) = L_{(x_0,y_0)}(x_1, y_1) - z_0$$

This quantity, dz is actually familiar to us and is known as a differential.

Definition

If f is a two-variable function such that f_x and f_y exist near (a, b) and are both continuous at (a, b), then f is differentiable at (a, b).

Definition

If f is a single-variable differentiable function, and $y = f(x)$, then as x changes from x_0 to some new value of x, the **differentials** are dx and dy, where

$$\begin{aligned} dx &= \Delta x \\ dy &= f'(x)\ dx \end{aligned}$$

In this case, dx represents the change in x, and dy represents the respective change in y on the tangent line to f at $(x_0, f(x_0))$.
If $z = f(x, y)$ is a differentiable function, then as (x, y) changes from (x_0, y_0) to some nearby point, the **differentials** are dx, dy, and dz, where

$$\begin{aligned} dx &= \Delta x \\ dy &= \Delta y \\ dz &= f_x(x, y)\ dx + f_y(x, y)\ dy \\ &= \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy \end{aligned}$$

We sometimes call dz the **total differential**.

Example 3. Let $f(x, y) = \cos(3x + 5y)$. Find the linearization of f at $(2, -1)$. Compare $f(1.9, -0.9)$ with $L_{(2,-1)}(1.9, -0.9)$. What is the approximate Δz that we get using this linearization?

Example 4. A box company makes closed rectangular boxes of size 80 cm, 60 cm, and 50 cm, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

Example 5. Data is gathered and is found to have the following relationship.

x \ y	6	9	12	15	18
8	12	15	17	16	16
10	17	21	24	24	23
12	21	25	28	30	31
14	24	25	29	33	35
16	26	27	28	28	30

There is no datum for when $(x, y) = (15, 8)$. Find a linear approximation for $z = f(x, y)$ when x is near 14 and y is near 9. Use this approximation to approximate $f(15, 8)$.

16 The Chain Rule

16.1 Single-variable Calculus

Recall the Chain Rule for single-variable functions – if $y = f(u)$, and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

For practice, try out this related rates problem.

Example 0. A thief is running for the door of a store at a speed of 15 feet per second. A bystander notices and decides to take action as she runs down a perpendicular aisle to tackle them at 18 feet per second. When the theif is 5 feet from the spot they'll be tackled and woman is 6 feet from that spot, how fast is the distance between them shrinking?

Begin by writing down the information that you *have* and the information that you *want*. Then try setting up an equation that relates these quantities. Decide what the independent variable is. Then differentiate the equation (either explicitly or implicitly), and solve for the desired quantity.

For more than one variable, we have different "rules" depending on if the function is presented explicitly with one parameter, explicitly with two (or more) parameters, and implicitly.

16.2 One Parameter

Theorem

The Chain Rule (Explicit with One Parameter): If $z = f(x, y)$ is differentiable with $x = g(t)$ and $y = h(t)$, then z is a differentiable function of t, and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 1. Chemists have determined that for a mole of an ideal gas, pressure P (kPa), volume V (L), and temperature T (K) are related by the equation $PV = 8.31T$. Find the rate at which the volume is changing when the pressure is 0.1 kPa and is decreasing at a rate of 0.05 kPa/s, and the temperature is 295 K and is increasing at a rate of 0.1 K/s.

Example 2. Let $z = x^2 \cos y$, where $x(t) = t^2 + 1$ and $y(t) = \ln t$. Find $\frac{dz}{dt}\big|_{t=1}$.

16.3 Two Parameters

Theorem

The Chain Rule (Explicit with Two Parameters: If $z = f(x, y)$ is differentiable with $x = g(s, t)$ and $y = h(s, t)$, then z is a differentiable function of s and t, and

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad , \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example 3. Let $z = x^2 \cos y$, where $x(t) = t^2 + s$ and $y(t) = se^t$. Find formulas for $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial s}$.

16.4 General Chain Rule (More Parameters)

Similar diagrams can be made for a single function with as many intermediate functions and as many parameters as you like. We can thus generalize this rule.

Theorem

The Chain Rule (General): Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, where each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a differentiable function of $t_1, t_2, \ldots, t_m$, and

$$\begin{aligned}\frac{\partial u}{\partial t_i} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i} \\ &= \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}\end{aligned}$$

for each $i = 1, 2, \ldots, m$.

Note: The intermediate functions $x_1, x_2, \ldots, x_n$ do not *all* have to involve $t_1, t_2, \ldots, t_m$; however, any parameter that any of the x_j happen to be a function of should be included in this list.

Moreover, if x_j does not have t_k as a parameter, then when is $\dfrac{\partial x_j}{\partial t_k}$?

Example 4. Suppose $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$. Find $\dfrac{\partial u}{\partial s}$ where $r = 2$, $s = 1$, and $t = 0$.

16.5 Implicit Differentiation

Just as in single-variable functions, we may not be able to (or want to!) isolate a particular variable to identify it as being dependent upon the others. In these cases, we can decide to move all quantities to a single side of an equation, leaving

$$F(x, y) = 0$$

In this case, we can use the chain rule to find $\frac{dy}{dx}$. That is,

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

But since $\frac{dx}{dx} = 1$, we can solve for $\frac{dy}{dx}$, assuming $\frac{\partial F}{\partial y} \neq 0$. The result is the Implicit Function Theorem.

Theorem

If F is a two-variable function defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then $F(x, y) = 0$ defines y as a function of x near (a, b), and

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Moreover, if $F(x, y, z) = 0$, then under similar conditions,

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example 5. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $yz = \ln(x + z)$.

17 The Gradient

17.1 Directional Derivatives

We've already looked at some directional derivatives. That is, $f_x(x_0, y_0)$ represents the derivative of f in the $\mathbf{i}$ direction at the point (x_0, y_0). Moreover, $f_y(x_0, y_0)$ represents the derivative of f in the $\mathbf{j}$ direction at the point (x_0, y_0). We can generalize this to find the derivative of f in *any* direction.

Consider the surface S with equation $z = f(x, y)$ and the point $P(x_0, y_0, z_0)$ on S. Let $\mathbf{u} = \langle a, b\rangle$ be any *unit* vector on the xy-plane.

Goal: Find the instantaneous rate of change of z in the direction of $\mathbf{u}$ at the point P.

GeoGebra link: https://www.geogebra.org/3d/u84jzsqh

Suppose $\mathbf{u}$ has the representation whose initial point is at the projection of P onto the xy-plane. That is, $\mathbf{u}$ has its initial point at $P'(x_0, y_0, 0)$.

Suppose now that P changes by an amount of h in the direction of $\mathbf{u}$ to a new point Q on the surface. Then

$$P(x_0, y_0, f(x_0, y_0)) \longrightarrow Q(x_0 + ha, y_0 + hb, f(x_0 + ha, y_0 + hb))$$

If $Q'(x_0 + ha, y_0 + hb, 0)$ is the projection of Q onto the xy-plane, then this change causes P' to change by an amount of h along $\mathbf{u}$ to the point Q'. That is

$$P'(x_0, y_0, 0) \longrightarrow Q'(x_0 + ha, y_0 + hb, 0)$$

Notice that $\Delta z = f(x_0 + ha, y_0 + hb) - f(x_0, y_0)$, and

$$\lim_{Q' \to P'} \frac{\Delta z}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

represents the rate of change of z with respect to distance in the direction of $\mathbf{u}$.

Definition

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

provided this limit exists. Note that this is a single-variable limit.

Example 1. Suppose f is a differentiable two-variable function. What does $D_{\mathbf{i}} f(x, y)$ represent? What does $D_{\mathbf{j}} f(x, y)$ represent?

Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \mathbf{u} = f_x(x,y)a + f_y(x,y)b$$

Proof:

Let $g(h) = f(x_0 + ha, y_0 + hb)$. Then

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} \\ &= \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0) - f(x_0, y_0)}{h} \\ &= D_{\mathbf{u}} f(x_0, y_0) \end{aligned}$$

Moreover, if $x = x_0 + ha$ and $y = y_0 + hb$, then $g(h) = f(x,y)$, $\frac{dx}{dh} = a$, $\frac{dy}{dh} = b$, and

$$\begin{aligned} g'(h) &= \frac{d}{dh}(f(x_0 + ha, y_0 + hb)) \\ &= \frac{d}{dh}(f(x,y)) \\ &= \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} \\ &= f_x(x,y)a + f_y(x,y)b \end{aligned}$$

It follows that

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) &= g'(0) \\ &= f_x(x_0, y_0)a + f_y(x_0, y_0)b \end{aligned}$$

□

Example 2. Let $f(x,y) = 3x^3 - x^2y - y^3$. Find the directional derivative $D_{\mathbf{u}} f(1,2)$ if $\mathbf{u}$ is the unit vector given by the angle $\theta = \frac{\pi}{3}$.

17.2 The Gradient Vector

Definition

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f (read "del f" or "grad f") defined by

$$\nabla f = \langle f_x(x,y), f_y(x,y)\rangle = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

Example 3. Let $f(x,y) = 3x^3 - x^2y - y^3$. Find ∇f.

Theorem

If f is a function of two variables x and y and $\mathbf{u}$ is any unit vector in $\mathbb{R}^2$, then

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

Example 4. Find the directional derivative of $f(x,y) = 3x^3 - x^2y - y^3$ at the point $(1,2)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$.

Definition

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of the unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}f(x_0,y_0,z_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$
$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h\to 0}\frac{f(\mathbf{x}_0+h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

provided the limit exists.

Definition

If f is a function of three variables x, y, and z, then the **gradient** of f is defined by the vector function ∇f defined by

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

and

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

where $\mathbf{u}$ is any unit vector in $\mathbb{R}^3$.

17.3 Maximizing the Directional Derivative

Theorem

If f is a differentiable function of two or three variables, then the maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$, and this value occurs when $\mathbf{u}$ has the same direction (is parallel to) as the gradient vector $\nabla f(\mathbf{x})$.

Proof:

Recall that $\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2| \cos\theta$, where θ is the smallest angle between $\mathbf{v}_1$ and $\mathbf{v}_2$. In this case,

$$\begin{aligned} D_{\mathbf{u}} f &= \nabla f \cdot \mathbf{u} \\ &= |\nabla f||\mathbf{u}| \cos\theta \\ &= |\nabla f| \cos\theta \end{aligned}$$

where θ is the smallest angle between ∇f and $\mathbf{u}$. Since $\cos\theta$ is maximized when $\theta = 0$. Therefore, the maximum of $D_{\mathbf{u}} f$ is $|\nabla f|$, and this value occurs when $\theta = 0$. Moreover, when $\theta = 0$, $\mathbf{u}$ has the same direction as ∇f.

□

Example 5. The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$, where T is measured in °C, and x, y, z in meters. In which direction does the temperature increase fastest at $P(2, -1, 2)$?

17.4 Tangent Planes to Level Surfaces

Suppose S is a level surface of $F(x, y, z)$ – that is, S is the graph of $F(x, y, z) = k$ for some $k \in \mathbb{R}$. Moreover, suppose $P(x_0, y_0, z_0)$ is a point on S and C is a curve that lies on S passing through P. It follows that C is described by a continuous function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. If t_0 is the parameter value that corresponds to P, then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since C is on S, any point $(x(t), y(t), z(t))$ must satisfy $F(x, y, z) = k$. Hence,

$$F(x(t), y(t), z(t)) = k$$

Now, if $x(t), y(t), z(t)$ are differentiable, then F is differentiable, and then

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\ &= \nabla F \cdot \mathbf{r}'(t) \end{aligned}$$

When $t = t_0$,

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

This last statement says that $\nabla F(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$.

Definition

Suppose $F(x, y, z)$ has level surface S corresponding to $F(x, y, z) = k$. If F is differentiable and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ is the plane passing through P with normal vector $\nabla F(x_0, y_0, z_0)$. The equation of this plane is

$$\begin{aligned} 0 &= \nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0\rangle \\ &= F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) \end{aligned}$$

The **normal line** to S at P is the line passing through P perpendicular to the tangent plane. The normal line has direction given by $\nabla F(x_0, y_0, z_0)$, and its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Theorem

If S is the surface whose equation is $z = f(x, y)$, then S is obtained from $F(x, y, z) = f(x, y) - z = 0$. It follows that the equation of the tangent plane to S at $P(x_0, y_0, z_0)$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + (-1)(z - z_0) = 0$$

Example 6. Find the equations of the tangent plane and normal line to $x - z = 4\arctan(yz)$ at $(1 + \pi, 1, 1)$.

18 Extrema

18.1 Definitions of Extrema

Definition

Suppose f is a function of two variables and has domain $\mathscr{D} \subseteq \mathbb{R}^2$. Then f has ...

- a **local maximum** at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) near (a, b), and the number $f(a, b)$ is the **local maximum value**.
- a **local minimum** at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) near (a, b), and $f(a, b)$ is called the **local minimum value**.
- an **absolute maximum** at (a, b) if $f(a, b) \geq f(x, y)$ for all $(x, y) \in \mathscr{D}$, and the number $f(a, b)$ is the **absolute maximum value**.
- an **absolute minimum** at (a, b) if $f(a, b) \leq f(x, y)$ for all $(x, y) \in \mathscr{D}$, and the number $f(a, b)$ is the **absolute minimum value**.

Definition

The word **extremum** means "maximum or minimum value". The word **extrema** is plural for extremum. That is, "Find the extrema of f" means "Find the maximum and minimum values of f."

Definition

Suppose f is a function of two variables. A point (a, b) is called a **critical point** (or **stationary point**) of f is $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or at least one of these values does not exist.

Theorem

Fermat's Theorem: If f has a local extremum at (a, b), then (a, b) is a critical point of f.

Caution: Just because (a, b) is a critical point of f does not mean that f has an extremum at (a, b).

Graphically, Fermat's Theorem states that if $f_x(a, b)$ and $f_y(a, b)$ exist, then the local extrema of f will have a *horizontal* tangent plane at (a, b).

Example 1. Compare the graphs of $f(x, y)$ and $g(x, y)$ at https://www.geogebra.org/3d/b9jmtgwk. Determine where the critical points of each function are. Determine where the extrema of each function are.

18.2 The Discriminant & the Second Derivatives Test

Definition

Suppose f is a two-variable function. We define the **discriminant** of f to be D, where

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

$$D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$

Theorem

Suppose the second partial derivatives of f are continuous near (a,b). Suppose further that $f_x(a,b) = f_y(a,b) = 0$. Let D be the discriminant of f.

- If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
- If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
- If $D < 0$, then $f(a,b)$ is not an extremum, but $(a,b,f(a,b))$ is called a **saddle point** of f.
- If $D = 0$, then the second derivative test is inconclusive.

Example 2. Let $f(x,y) = x^3 - 3x + 3xy^2$. Find the local extrema and the saddle point(s) of f. Use GeoGebra to check your conclusion.

18.3 Absolute Extrema

If f is a single-variable function defined on a closed interval, then the Extreme Value Theorem guarantees that f attains both absolute extrema on that closed interval.

To extend the Extreme Value Theorem to two variables, we must also extend the concept of a closed interval to two variables.

Definition

Let $\mathscr{D}$ be a set in $\mathbb{R}^2$.

- A **boundary point** of $\mathscr{D}$ is a point $(a, b) \in \mathscr{D}$ such that every disk centered at (a, b) will contain both points inside of $\mathscr{D}$ and outside of $\mathscr{D}$.
- The set $\mathscr{D}$ is a **closed set** if it contains all of its boundary points.
- The set $\mathscr{D}$ is an **open set** if it contains none of its boundary points.
- The set $\mathscr{D}$ is a **bounded set** if it is contained within some disk of finite radius.

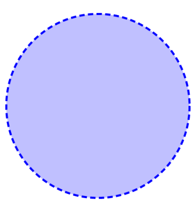


Figure 4: Open Set

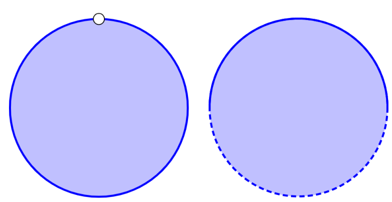


Figure 5: Neither Closed Nor Open Set

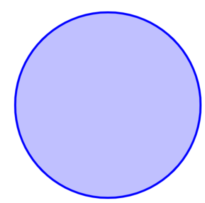


Figure 6: Closed Set

Theorem

The Extreme Value Theorem for Functions of Two Variables: If f is continuous on a closed, bounded set $\mathscr{D} \subset \mathbb{R}^2$, then f attains both absolute extrema at some points in $\mathscr{D}$.

Strategy

To find the absolute extrema of a continuous function f on a closed, bounded set $\mathscr{D}$:

1. Find the critical points of f in $\mathscr{D}$.
2. Evaluate f at the critical points of f.
3. Find the extreme values of f on the boundary of $\mathscr{D}$.
4. The largest value found in steps (2) and (3) is the absolute maximum of f on $\mathscr{D}$. The smallest value is the absolute minimum.

Example 3. Find the absolute extrema of $f(x, y) = x^2 + xy + y^2 - 6y$ on the set $\mathscr{D}$, where $\mathscr{D} = \{(x, y) \mid -3 \leq x \leq 3, y \in [0, 5]\}$.

Example 4. Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point $(0, 0, 1)$.

19 Lagrange Multipliers

Exploration: Suppose we want to find the extrema of $f(x, y) = x^2 + 2y^2$ but also know that any (x, y) must be on the unit circle. Graphically, $z = f(x, y)$ produces a paraboloid whose vertex is at the origin and opens in the direction of the positive z-axis.

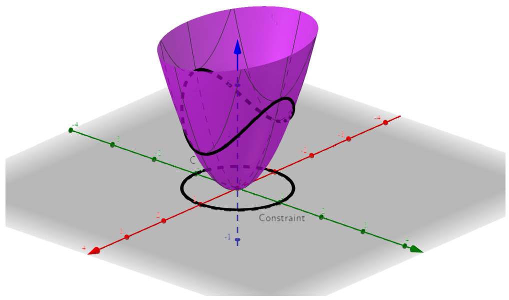


Figure 7: https://www.geogebra.org/3d/kmw4uhnw

Now, to find the maximum of $f(x, y) = x^2 + 2y^2$ subject to $x^2 + y^2 = 1$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $x^2 + y^2 = 1$. For convenience, let's call $g(x, y) = x^2 + y^2$. Now, this happens specifically when these curves just touch each other; this happens specifically when the curves share a tangent line; this happens specifically when the normal lines are parallel; this happen specifically when $\nabla f(x_0, y_0) = \lambda \nabla g(x, y)$ for some $\lambda \in \mathbb{R}$.

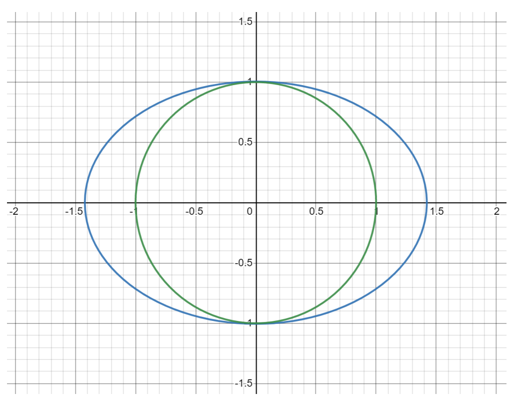


Figure 8: https://www.desmos.com/calculator/wenqh2aldm

To formalize this, suppose a function f has an extremum at $P(x_0, y_0, z_0)$ on the surface S, and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ that lies on S and passes through P. Further suppose $t_0 \in \mathbb{R}$ such that $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Then $h(t) = f((x(t), y(t), z(t))$ represents the values of f that lie on C. Since f has an extremum at (x_0, y_0, z_0), h must have an extremum at t_0, so by Fermat's Theorem, $h'(t_0) = 0$. Moreover,

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

Thus, $\nabla f(x_0, y_0, z_0)$ and $\mathbf{r}'(t_0)$ must be orthogonal. Since we know that the gradient vector at P is perpendicular to the tangent vector to any curve C on S at P, we know that $\nabla g(x_0, y_0, z_0)$ is also orthogonal to $\mathbf{r}'(t_0)$. Since both $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are orthogonal to $\mathbf{r}'(t_0$, it follows that $\nabla f(x_0, y_0, z_0)$ is parallel to $\nabla g(x_0, y_0, z_0)$. Therefore, if $\nabla g(x_0, y_0, z_0) \neq 0$, then there must exist a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

Definition

The number λ such that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$ is called a **Lagrange multiplier**.

Method of Lagrange Multipliers

To optimize a multivariable function $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming extrema exist),

1. Find all values of x, y, z, λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \qquad \text{and} \qquad g(x, y, z) = k$$

2. Evaluate f at all of the point (x, y, z) that you found in the previous step. The largest of these values is the maximum of f, and the least value is the minimum of f.

Note: Solving for x, y, z, λ is not necessarily straightforward. In three variables, solving $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ produces four equations in four unknowns. That is,

$$f_x = \lambda g_x \tag{1}$$
$$f_y = \lambda g_y \tag{2}$$
$$f_z = \lambda g_z \tag{3}$$
$$g = k \tag{4}$$

In two variables, solving $\nabla f(x, y) = \lambda \nabla g(x, y)$ produces three equations in three unknowns. That is,

$$f_x = \lambda g_x \tag{1}$$
$$f_y = \lambda g_y \tag{2}$$
$$g = k \tag{3}$$

The kicker is that these are *not necessarily linear systems*. Linear algebra studies solving linear systems, but we very well may have a nonlinear system, so you'll have to use some ingenuity.

Example 1. Find the extrema of $f(x, y) = x^2 + 2y^2$ on the unit circle.

Example 2. Find the extrema of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.