MTH 254 Guided Notes

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Contents

8 Vector Functions

Definition

A **vector-valued function** or **vector function** is a function whose domain is a subset of $\mathbb{R}$ and whose range is a subset of vectors. In particular, we will typically deal with $\mathbf{r}$, whose domain is a subset of $\mathbb{R}$ and whose range is a subset of V_3. If

$$\begin{aligned}\mathbf{r}(t) &= \langle f(t), g(t), h(t)\rangle \\ &= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}\end{aligned}$$

then we say that f, g, and h are the **component functions** of $\mathbf{r}$.

Example 1. Find the domain of $\mathbf{r}(t) = \langle \ln(7-t), \sqrt[3]{t}, \sqrt{t+1}\rangle$.

Definition

The **limit** of a vector function $\mathbf{r}$ is defined by taking the limit of its component functions. That is, if $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

provided the limits of the component functions exist .

Example 2. Let $\mathbf{r}(t) = \langle \ln(7-t), \sqrt[3]{t}, \sqrt{t+1}\rangle$. Find $\lim\limits_{t\to 0} \mathbf{r}(t)$.

Definition

A vector function $\mathbf{r}$ is **continuous at** a if $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

8.1 Graphing Parametric Equations

Definition

Suppose f, g, and h are continuous real-valued functions on an interval I. Then the set C of all (x, y, z) in space, where

$$\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned} \tag{1}$$

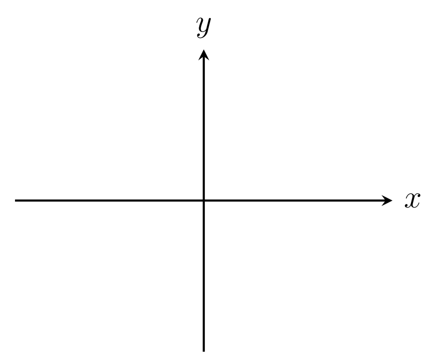
with $t \in I$ is called a **space curve**. The equations (1) are called the **parametric equations of** C, and t is the **parameter**.

Notes: When sketching parametric equations,

- Draw and label your axes,
- Draw and label tick marks to show scale,
- Plot some points (tables are sometimes nice),
- Sketch the curve, and
- Draw arrows on the curve to depict which *direction* the curve moves.

Example 3. Sketch the curve whose vector equation is given below. Then use `GeoGebra.org` to plot your graph.

$$\mathbf{r}(t) = \langle \cos t, t^2 \rangle$$



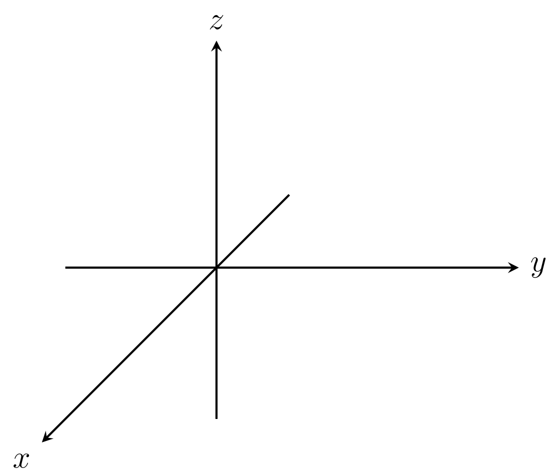
Note: To plot a parametric vector function in GeoGebra, use the command

```
r(t) = ( f(t) , g(t) )
```

where $x = f(t)$ and $y = g(t)$.

Example 4. Sketch the curve whose vector equation is given below. Then use www.GeoGebra.org to plot your graph.

$$\mathbf{r}(t) = t\mathbf{i} + \sin(\pi t)\mathbf{j} - \cos(\pi t)\mathbf{k}$$



Note: To plot a three-dimensional parametric vector function in GeoGebra, use the command

```
r(t) = ( f(t) , g(t) , h(t) )
```

where $x = f(t)$, $y = g(t)$, and $z = h(t)$. GeoGebra will automatically convert your command into

```
r = Curve( ( f(t) , g(t) , h(t) ), t, START, END)
```

This command identifies `r` as the name of your function, `t` as the parameter, `START` as the beginning value of your parameter, and `END` as the end value of your parameter. That is, the domain of your function is [`START`, `END`].

If you want to see the *motion* of the curve, use the command

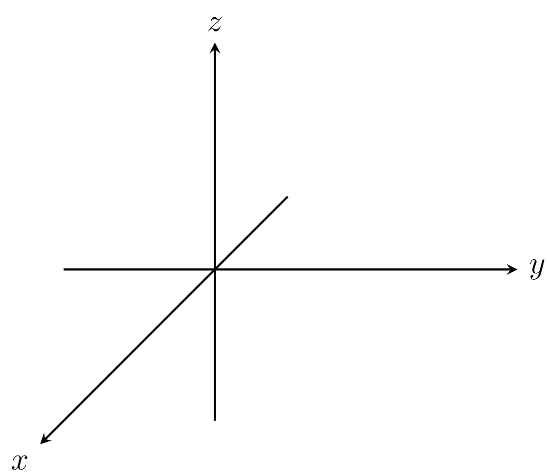
```
P = r(k)
```

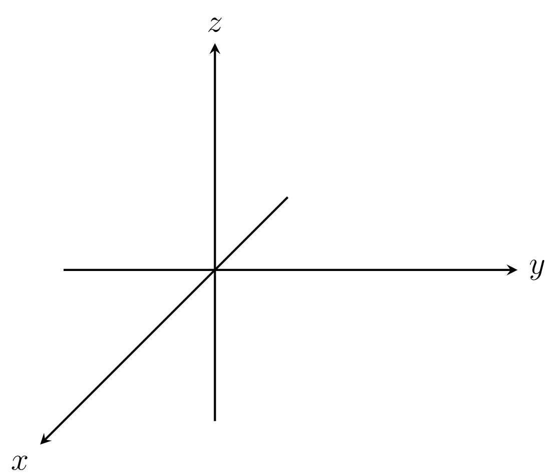
to create a point named P. This will also create a slider for k, which you can move around to see the point move along the curve. As you increase k, you will move along the curve in the positive direction.

8.2 Intersecting Surfaces

The intersection of two curves in $\mathbb{R}^2$ is typically a small collection of points. The intersection of surfaces in $\mathbb{R}^3$ is typically a curve. Our goal will be to identify this curve and describe it parametrically.

Example 5. Sketch the surfaces $x^2 + y^2 = 4$ and $z = x^2$ by hand on the axes below. Then, analytically find a vector function that represents the intersection of the two surfaces. Check your work by plotting the two surfaces in GeoGebra as well as the vector function.





Example 6. Find the point(s) of intersection of the curve $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ with the surface $x^2 + y^2 + z^2 = 17$. Check your conclusion in GeoGebra.

9 Calculus of Vector Functions

9.1 Tangent Vectors & Tangent Lines

Definition

The **derivative**, $\mathbf{r}'$, of a vector function $\mathbf{r}$ is defined just as for single-variable functions:

$$\begin{aligned}\mathbf{r}'(t) &= \frac{d\mathbf{r}}{dt}\\ &= \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}\end{aligned}$$

provided the limit exists.

Remark: If $y = f(x)$, then $\frac{dy}{dx}$ represents the *slope* of the *tangent line* to the curve at any given point. Here, if $\mathbf{r}(t)$ is a vector function, then $\mathbf{r}'(t)$ represents the *tangent vector* to the curve at any given point.

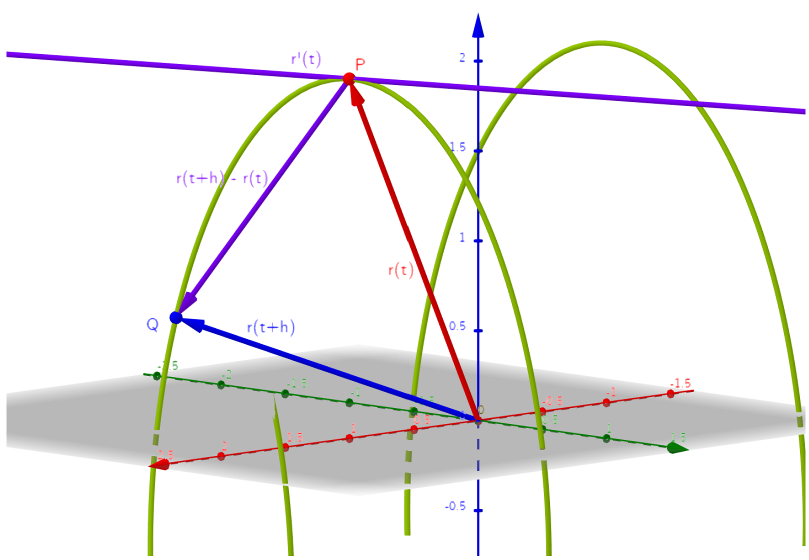


Figure 1: https://www.geogebra.org/3d/rdghwbxk

Definition

Let C be the curve produced by the vector function $\mathbf{r}(t)$ and P be the point on C at when $t = a$. The vector $\mathbf{r}'(a)$ is called a **tangent vector** to the curve C at the point P, provided $\mathbf{r}'(a)$ exists and $\mathbf{r}'(a) \neq \mathbf{0}$. A **unit tangent vector** is a tangent vector to C at P whose length is 1. To obtain a unit tangent vector $\mathbf{T}(t)$, we normalize $\mathbf{r}'(t)$:

$$\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The **tangent line** to C at P is the line through P parallel to $\mathbf{r}'(a)$.

Theorem

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, h are differentiable functions, then

$$\begin{aligned}\mathbf{r}'(t) &= \langle f'(t), g'(t), h'(t)\rangle \\ &= f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}\end{aligned}$$

Proof:

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}(\mathbf{r}'(t+\Delta t) - \mathbf{r}(t)) \\ &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left(\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)\rangle\right) \\ &= \lim_{\Delta t\to 0} \left\langle \frac{f(t+\Delta t)}{\Delta t}, \frac{g(t+\Delta t)}{\Delta t}, \frac{h(t+\Delta t)}{\Delta t}\right\rangle \\ &= \left\langle \lim_{\Delta t\to 0}\frac{f(t+\Delta t)}{\Delta t}, \lim_{\Delta t\to 0}\frac{g(t+\Delta t)}{\Delta t}, \lim_{\Delta t\to 0}\frac{h(t+\Delta t)}{\Delta t}\right\rangle \\ &= \langle f'(t), g'(t), h'(t)\rangle\end{aligned}$$

□

Example 1. Given the vector function $\mathbf{r}(t) = (t-2)\mathbf{i} + (t^2+1)\mathbf{j}$, complete the following:

i. Draw and label a set of Cartesian coordinate axes with an appropriate scale.

ii. Sketch $\mathbf{r}(t)$.

iii. Sketch the position vectors $\mathbf{r}(-2), \mathbf{r}(-1)$, and $\mathbf{r}(0)$.

iv. Sketch the tangent vector $\mathbf{r}'(-1)$.

v. Determine the equation of the tangent line at $t = -1$.

Example 2. Let $\mathbf{r}(t) = \langle t, \sin(\pi t), -\cos(\pi t)\rangle$.

i. Find $\mathbf{r}'(t)$.

ii. Find $\mathbf{T}(t)$.

iii. Find the equation of the tangent line to $\mathbf{r}(t)$ at the point where $t = 1$.

iv. Use GeoGebra to plot $\mathbf{r}(t)$, $\mathbf{r}'(1)$, $\mathbf{T}(1)$, and the line tangent to $\mathbf{r}(t)$ at the point where $t = 1$.

9.2 Rules of Differentiation

Multiple derivatives work in the same way with vector-valued functions. For example,

$$\mathbf{r}''(t) = \frac{d}{dt}\frac{d\mathbf{r}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

and

$$\mathbf{r}''(t) = \langle f''(t), g''(t), h''(t)\rangle$$
$$\mathbf{r}^{(n)}(t) = \langle f^{(n)}(t), g^{(n)}(t), h^{(n)}(t)\rangle$$

Theorem

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, $c \in \mathbb{R}$, and f is a real-valued function. Then

i. $\frac{d}{dt}\left(\mathbf{u}(t) + \mathbf{v}(t)\right) = \mathbf{u}'(t) + \mathbf{v}'(t)$

ii. $\frac{d}{dt}\left(c\mathbf{u}(t)\right) = c\mathbf{u}'(t)$

iii. $\frac{d}{dt}\left(f(t)\mathbf{u}(t)\right) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

iv. $\frac{d}{dt}\left(\mathbf{u}(t) \cdot \mathbf{v}(t)\right) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

v. $\frac{d}{dt}\left(\mathbf{u}(t) \times \mathbf{v}(t)\right) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

vi. (Chain Rule): $\frac{d}{dt}\left(\mathbf{u}(f(t))\right) = f'(t)\mathbf{u}'(f(t))$

Example 3. Let $\mathbf{r}_1(t) = \langle t, \sin(\pi t), -\cos(\pi t)\rangle$ and $\mathbf{r}_2(t) = e^{-t}\mathbf{i} + 6t\mathbf{j} + (t\ln t)\mathbf{k}$.

i. Find $\mathbf{r}_1''(t)$.

ii. Find $\frac{d}{dt}\left(t^2\mathbf{r}_1(t) + \mathbf{r}_1(t^2)\right)$.

iii. Find $\frac{d}{dt}\left(\mathbf{r}_1(t) \times \mathbf{r}_2(t)\right)$ in two ways: first by finding the cross product and then differentiating the result, and second by using the formula from Theorem section 9.2.

9.3 Integrals of Vector Functions

Definition

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ is defined just as for single-variable functions:

$$\begin{aligned}\int_a^b \mathbf{r}(t)\ dt &= \lim_{n\to\infty}\sum_{i=1}^{n}\mathbf{r}(t_i^*)\Delta t\\ &= \lim_{n\to\infty}\left\langle \sum_{i=1}^{n} f(t_i^*)\Delta t, \sum_{i=1}^{n} g(t_i^*)\Delta t, \sum_{i=1}^{n} h(t_i^*)\Delta t\right\rangle\\ &= \left\langle \int_a^b f(t)\ dt, \int_a^b g(t)\ dt, \int_a^b h(t)\ dt\right\rangle\\ &= \mathbf{R}(t)\Big|_a^b \qquad \text{(FTC 2)}\end{aligned}$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$ on an interval containing $[a,b]$; that is, $\mathbf{R}'(t)=\mathbf{r}(t)$.

We use the notation $\int \mathbf{r}(t)\ dt$ to indicate the **indefinite integral** of the vector function $\mathbf{r}(t)$.

Example 4. Find $\displaystyle\int_1^{\sqrt{3}}\left(\frac{2}{1+t^2}\mathbf{i}-\frac{2t}{1+t^2}\mathbf{j}\right)\ dt$.

Example 5. Let $\mathbf{r}(t)=\langle t, \sin(\pi t), -\cos(\pi t)\rangle$.

i. Find $\displaystyle\int \mathbf{r}(t)\ dt$.

ii. Find $\displaystyle\int_0^1 \mathbf{r}(t)\ dt$.

10 Arc Length and Curvature

10.1 Arc Length in Space

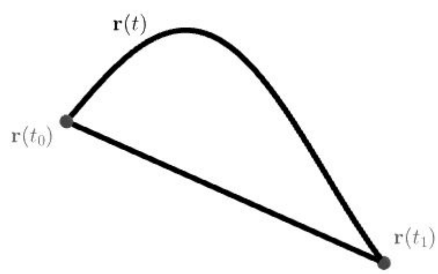
Recall that if $\mathbf{u} = \langle u_1, u_2.u_3\rangle$, then $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$|\mathbf{r}(t)| = \sqrt{[f(t)]^2 + [g(t)]^2 + [h(t)]^2}$$

and so

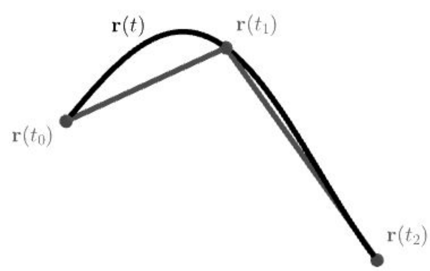
$$|\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Consider a curve C given by $\mathbf{r}(t)$ in $\mathbb{R}^3$.



Suppose the endpoints on C are $\mathbf{r}(t_0)$ and $\mathbf{r}(t_1)$. Then the distance between $\mathbf{r}(t_0)$ and $\mathbf{r}(t_1)$, D, is an approximation for the length of C and is given by

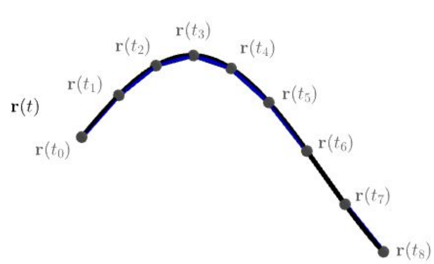
$$D = |\mathbf{r}(t_1) - \mathbf{r}(t_0)|$$



Suppose the endpoints on C are $\mathbf{r}(t_0)$ and $\mathbf{r}(t_2)$. We can then include one more point on C, $\mathbf{r}(t_1)$, to improve our approximation for the length of C. Then the distances between successive points, D, is an approximation for the length of C and is given by

$$\begin{aligned} D &= |\mathbf{r}(t_1) - \mathbf{r}(t_0)| + |\mathbf{r}(t_2) - \mathbf{r}(t_1)| \\ &= \sum_{i=1}^{2} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| \end{aligned}$$

If we continue this a la differential and integral calculus, then we can see a similar limit pattern emerge. Our approximations will more resemble our curve, and we can form an integral.



Suppose the endpoints on C are $\mathbf{r}(t_0)$ and $\mathbf{r}(t_8)$. Then the distances between successive points, D, is an approximation for the length of C and is given by

$$D = \sum_{i=1}^{8} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|$$

Theorem

Suppose a curve in 3-space has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where $t \in [a, b]$ and f', g', h' are continuous on $[a, b]$. If the curve is traversed exactly once as t increases from a to b, then the length of the curve L is

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \int_a^b |\mathbf{r}'(t)|\, dt \end{aligned}$$

Example 1. Find the length of the curve whose vector equation is $\mathbf{r}(t) = \langle t, \sin(\pi t), \cos(\pi t)\rangle$ over the interval $0 \le t \le 2$.

Suppose we allow the upper limit of the arc length formula to vary. This allows us to define a new function that inputs the upper limit of the parameter and outputs the length of the arc.

Definition

Let C be a curve whose vector function is $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $t \in [a, b]$, $\mathbf{r}'(t)$ is continuous on $[a, b]$, and where C is traversed exactly once as t increases from a to b. We define the **arc length function** s by

$$\begin{aligned} s(t) &= \int_a^t |\mathbf{r}'(u)|\, du \\ &= \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \end{aligned}$$

Thus, $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$.

Using FTC1, we can differentiate $s(t) = \int_a^t |\mathbf{r}'(u)|\, du$ with respect to t to get

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

10.2 Parametrization

Definition

Let $\mathbf{r}_1(t)$ be a vector equation of a curve C with $a \leq t \leq b$. If $\mathbf{r}_2(u)$ is another vector equation of C with $c \leq u \leq d$, then we say that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(u)$ are **parametrizations** of C. That is, $\mathbf{r}_1(t)$ and $\mathbf{r}_2(u)$ are two different parametric equations that produce exactly the same curve (though traversing these curves may occur at different paces or in a different orientation).

Suppose we are given a curve C whose vector equation may be

$$\mathbf{r}_1(t) = \langle t, 2t, t^2 + 1 \rangle \qquad 1 \leq t \leq 2$$

Now, if we substitute $t = e^u$, then we can rewrite our vector equation as

$$\mathbf{r}_2(u) = \left\langle e^u, 2e^u, e^{2u} + 1 \right\rangle \qquad 0 \leq u \leq \ln 2$$

Then $\mathbf{r}_1(t)$ and $\mathbf{r}_2(u)$ are different parametrizations of C.

Note: To see different parametrizations of a curve, we can use GeoGebra. First, define a curve

```
r = Curve( ( f(t) , g(t) , h(t) ), t, START, END)
```

Remember, this command identifies `t` as the parameter, `START` as the beginning value of your parameter, and `END` as the end value of your parameter. If you want to see the *motion* of the curve, use the command

```
P = r(k)
```

to create a point named P. This will also create a slider for k, which you can move around to see the point move along the curve. As you increase k, you will move along the curve in the positive direction.

To introduce a different parametrization, we can name a new point

```
Q = r( s(u) )
```

where $k = s(u)$ is your new parametrization. Make sure to monitor the upper and lower bounds of your parameters!

It is often quite advantageous to parametrize a curve with respect to arc length, because arc length arises naturally from the geometry of the curve rather than the algebra we use to describe the curve.

Example 2. Reparametrize the curve $\mathbf{r}(t) = \langle t, \sin(\pi t), \cos(\pi t)\rangle$ with respect to arc length with $0 \leq t \leq 2$. Check both parametrizations in GeoGebra.

Definition

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$ on I. A curve C is called **smooth** if it has a smooth parametrization. Smooth curves are continuous with no cusps or corners; tangents turn continuously.

10.3 Curvature

If C is smooth and is defined by the vector function $\mathbf{r}$, then its unit tangent is $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. This indicates the direction that the curve turns.

Definition

If $\mathbf{r}(t)$ is a differentiable vector function, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

is called the **unit tangent function**. This function takes the parameter t as an input and outputs a unit vector pointing in the direction that the curve determined by $\mathbf{r}$ points.

Observations:

- When C is straight, $\mathbf{T}(t)$ does not change.
- When C is fairly straight, $\mathbf{T}(t)$ changes direction very slowly.
- When C is quite curved, $\mathbf{T}(t)$ changes rapidly.

That is, how fast $\mathbf{T}(t)$ changes describes how quickly C changes direction. This is a derivative relationship, but we do not want this relationship to depend upon our parametrization – we therefore use s as our variable so that we can have independence from whatever parametrization is chosen.

Definition

The **curvature** of a curve C is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector to C.

Remark: That is, we can say that *curvature is the size of the rate that the unit tangent vector changes*. Since the unit tangent vector doesn't change size, this is only a measure of how fast the *direction* the tangent vector changes.

Computationally, it is easier to compute with an original parameter t. By the Chain Rule,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \text{so} \qquad \kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} \right|$$

Because $\frac{ds}{dt} = |\mathbf{r}'(t)|$, we can rewrite curvature as

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

There are other formulas, as well. One such formula is

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Example 3. Find the curvature of a planar circle of radius a.

Example 4. Find the curvature of $\mathbf{r}(t) = \langle t^2, \ln t, t \ln t\rangle$ at the point $(1, 0, 0)$. Explain as specifically as you can what this number represents.

10.4 Unit Tangent and Unit Normal Vectors

Given the differentiable vector function $\mathbf{r}(t)$, we have defined the unit tangent function as $\mathbf{T}(t)$, which produces a unit tangent vector at the point corresponding to t.

If we observe the same thought to a unit tangent vector, then we make the computation $\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. This produces a vector that measures the change in $\mathbf{T}(t)$ over its magnitude. Since a unit tangent vector does not change size, then this vector must account only for the change in direction.

Definition

Suppose $\mathbf{r}(t)$ is the vector function for a smooth curve. A **unit normal vector** is a unit vector that is orthogonal to a unit tangent vector. The **principal unit normal vector** (or **unit normal**) is the unit normal vector that points in the direction that the curve is turning. In particular, the **unit normal function** is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

and produces the principal unit normal vector to the curve at the point determined by t. A unit normal is defined as long as $\kappa \neq 0$.

Convention: We will refer to unit tangent and unit normal vectors as those that are pointing in the direction that the curve is moving and turning as t increases.

For reference, see https://www.geogebra.org/m/NCTTCnUX.

Example 5. Let $\mathbf{r}(t) = t\mathbf{i} + \sin(\pi t)\mathbf{j} + \cos(\pi t)\mathbf{k}$, C be the curve determined by $\mathbf{r}(t)$, and P be the point $(1, 0, -1)$. Find the following.

a. The unit tangent vector vector to C at P.

b. The unit normal vector to C at P.

c. The curvature of C at P.

10.5 Binormals and Normal Planes

Definition

The **normal plane** to a curve C at a point P is comprised by all of the lines orthogonal to a tangent vector at P.

In order to find a plane, we only need two vectors and a point. Our point is P. One of those vectors is $\mathbb{N}$. The last vector we will use is called a binormal vector.

Definition

Suppose C is a smooth curve determined by $\mathbf{r}(t)$. A **binormal vector** to C at a point P is a unit vector perpendicular to both a tangent vector and a normal vector at P. The **binormal vector function** is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

and produces a binormal vector at the point corresponding to t.

Example 6. Let $\mathbf{r}(t) = t\mathbf{i} + \sin(\pi t)\mathbf{j} + \cos(\pi t)\mathbf{k}$, C be the curve determined by $\mathbf{r}(t)$, and P be the point $(1, 0, -1)$. Find an equation for the normal plane to the curve C at the point P.

We end with a theorem summarizing the formulas introduced in this section.

Theorem

If a smooth curve C given by $\mathbf{r}$, then

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \\
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
\kappa &= \left|\frac{d\mathbf{T}}{ds}\right| \\
\kappa &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \\
&= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} && \text{useful for computation} \\
\kappa(x) &= \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}} && \text{when } C \text{ is a planar curve with } y = f(x)
\end{aligned}$$

11 Velocity and Acceleration

Definition

Suppose $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a vector function that represents the position of an object. Then the **velocity vector** for that object at time t is given by $\mathbf{v}(t)$, and the **acceleration vector** for that object at time t is given by $\mathbf{a}(t)$, where

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t)\\ &= \left\langle \frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}\right\rangle\end{aligned} \qquad \begin{aligned}\mathbf{a}(t) &= \mathbf{v}'(t)\\ &= \left\langle \frac{d^2f}{dt^2}, \frac{d^2g}{dt^2}, \frac{d^2h}{dt^2}\right\rangle\end{aligned}$$

Moreover, the **speed** of the object at time t is given by

$$\begin{aligned}|\mathbf{v}(t)| &= |\mathbf{r}'(t)|\\ &= \frac{ds}{dt}\end{aligned}$$

Example 1. Let $\mathbf{r}(t) = \langle t, \sin(\pi t), \cos(\pi t)\rangle$ represent the position of a particle in space measured in meters after t seconds. Find the following.

a. The velocity vector for the particle after 2 seconds.

b. The speed of the particle after 2 seconds.

c. The acceleration vector for the particle after 2 seconds.

Example 2. Suppose a particle is moving such that its position is given by

$$\mathbf{r}(t) = (1-t)\mathbf{i} + 2\cos\left(\frac{\pi}{2}t\right)\mathbf{j}$$

a. Find $\mathbf{v}(t)$.

b. Find $\mathbf{a}(t)$.

c. Draw a set of labeled and scaled coordinate axes. Plot $\mathbf{r}(t)$ with $t \in [0, 5]$ and indicate the direction that $\mathbf{r}$ moves.

d. Plot the position, velocity, and acceleration vectors corresponding to when $t = 2$. The velocity and acceleration vectors should have their initial points at the terminal point of the position vector.

e. Find the speed of the particle at the point $(-2, 0)$.

11.1 Initial Value Problems

Example 3. A moving particle starts at an initial position $\mathbf{r}(0) = \mathbf{i}$, an initial velocity of $\mathbf{v}(0) = 2\mathbf{i} + \mathbf{j}$, and acceleration given by

$$\mathbf{a}(t) = \left\langle 4e^{2t}, \frac{-1}{(t+1)^2}, 24t^2 \right\rangle$$

Find the exact position vector for the particle when $t = 1$.

11.2 Contextual Initial Value Problems

Theorem

If a projectile is fired with angle of elevation α and initial velocity vector $\mathbf{v}_0$, and if we assume that air resistance is negligible and the only force acting on the object is gravity, then

$$\mathbf{r}(t) = \left\langle (v_0 \cos\alpha)t, (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \right\rangle + \mathbf{r}_0 \qquad \text{Position after } t \text{ seconds}$$

$$\mathbf{v}(t) = \langle v_0 \cos\alpha, v_0 \sin\alpha - gt \rangle \qquad \text{Velocity after } t \text{ seconds}$$

$$\mathbf{a}(t) = \langle 0, -g \rangle \qquad \text{Acceleration after } t \text{ seconds}$$

$$\mathbf{v}_0 = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle \qquad \text{Initial Velocity (at 0 seconds)}$$

where $v_0 = |\mathbf{v}_0|$ is the initial speed of the projectile, $\mathbf{r}_0$ is the initial position of the projectile, and g is the gravitational constant.

Note: We can use the values in place of g for a gravitational constant. These are not exact values, but these are the values that we typically use at sea level on Earth.

$$g \approx \begin{cases} 32 \ \frac{\text{ft}}{\text{s}^2} & \text{If measuring in Imperial units} \\ 9.8 \ \frac{\text{m}}{\text{s}^2} & \text{If measuring in metric units} \end{cases}$$

Example 4. Alison and Amanda have built a robot that launches small projectiles at an angle of elevation of 50° and initial velocity of 10 m/s relative to the direction it is pointing. If the robot is on a flat, horizontal plane, we assume negligible conditions, and the launcher is 1 meter off the ground, how far away from the robot will the projectile first hit the ground? Round your conclusion to the nearest hundredth of a meter.

Example 5. The airspeed velocity of an unladen African swallow is 11 m/s. This particular swallow is flying at 80.4 m above the ground, parallel to it, and is holding a 1.4 kg coconut (this swallow is particularly strong). There is a westward wind applying a steady 7 N to objects in the air. Arthur is 2 meters tall and is standing on the ground. If flying east at 11 m/s and drops the coconut, then how far east or west of Arthur does the swallow need to be to ensure that the coconut hits Arthur on the head?

11.3 Tangential and Normal Components of Acceleration

Recall that, given $\mathbf{r}(t)$, we can construct

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbb{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \tag{2}$$

If we suppose that $\mathbf{r}(t)$ represents motion of an object in space, then letting $v(t) = |\mathbf{v}(t)|$, we get

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{v(t)}$$

It follows that

$$\mathbf{v}(t) = v(t)\mathbf{T}(t) \tag{3}$$

Differentiating (3), we get

$$\mathbf{a}(t) = \mathbf{v}'(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) \tag{4}$$

From curvature, obtain

$$\kappa = \frac{|\mathbf{T}'(t)|}{v(t)} \qquad \Longleftrightarrow \qquad |\mathbf{T}'(t)| = \kappa v(t) \tag{5}$$

From the definition of the unit normal, substitute in (5) to get

$$\mathbf{T}'(t) = |\mathbf{T}'(t)|\mathbb{N}(t) \qquad \Longleftrightarrow \qquad \mathbf{T}'(t) = \kappa v(t)\mathbb{N}(t) \tag{6}$$

Looking again at (4), we get the following theorem.

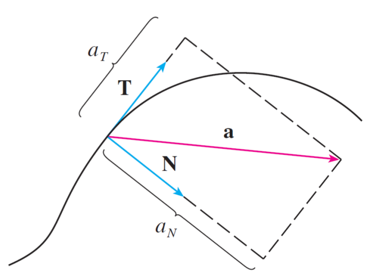
Theorem

If we suppose that $\mathbf{r}(t)$ represents motion of an object in space, then

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + \kappa[v(t)]^2\mathbb{N}(t) \qquad \text{or} \qquad \mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbb{N}$$

where $v(t) = |\mathbf{v}(t)|$. If we let $a_T = v'(t)$ and $a_N = \kappa[v(t)]^2$, then

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbb{N}$$



Definition

The quantities a_T and a_N described in the previous theorem are called the **tangential component of acceleration** and **normal component of acceleration**, respectively. These represent the scalars in the decomposition of $\mathbf{a}$ into a weighted sum of $\mathbf{T}$ and $\mathbb{N}$.

For more computational exploration, consider

$$\begin{aligned}\mathbf{v}\cdot\mathbf{a} &= v\mathbf{T}\cdot(v'\mathbf{T}+\kappa v^2\mathbb{N})\\ &= vv'(\mathbf{T}\cdot\mathbf{T})+\kappa v^3(\mathbf{T}\cdot\mathbb{N})\\ &= vv'\|\mathbf{T}\|^2+\kappa v^3(0)\\ &= vv'\end{aligned}$$

It follows that

$$\begin{aligned}a_T &= v'\\ &= \frac{\mathbf{v}\cdot\mathbf{a}}{v}\\ &= \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}\end{aligned}$$

Note that

$$v^2 = |\mathbf{v}(t)|^2 = |\mathbf{r}'(t)|^2$$

From (2), we can rewrite the normal component of acceleration as

$$\begin{aligned}a_N &= \kappa v^2\\ &= \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2\\ &= \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}\end{aligned}$$

Thus,

Corollary

If we suppose that $\mathbf{r}(t)$ represents motion of an object in space, then

$$\begin{aligned}a_T &= \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}\\ a_N &= \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}\end{aligned}$$

Example 6. Suppose a particle moves with position function $\mathbf{r}(t) = \langle t, \sin(\pi t), \cos(\pi t)\rangle$. Find the tangential and normal components of acceleration.

Example 7. Suppose a particle moves with position function $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. Find the acceleration of the particle at the point $(1, 1, 1)$ and decompose it into terms of its unit tangent and unit normal vectors.