MTH 254 Guided Notes

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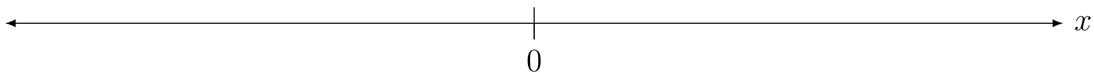
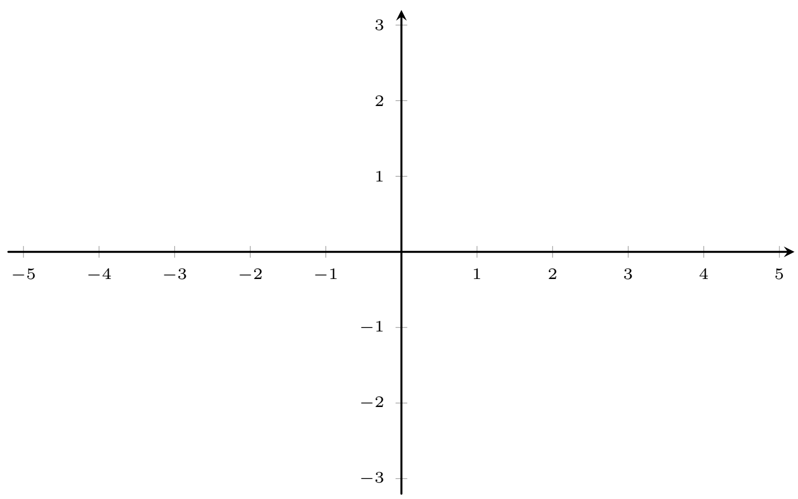
Contents

1 Introduction to 3D Coordinates

Exploration: To graph a single number, we need one line. The **Real Number Line** is drawn by beginning with a point O called the **Origin**, and a single **axis** (line) drawn on it.



To graph the Cartesian plane, we need two real number lines that intersect perpendicularly at the origin.



To graph in three dimensions, we need three real number lines that intersect perpendicularly at the origin.

Definition

To create a **3-dimensional Rectangular Coordinate System**, also known as **space**, **3-space**, or $\mathbb{R}^3$, we begin with a point O called the **Origin**. From O, we draw three directed perpendicular lines called the **coordinate axes**, labelling them the x-, y-, and z-axes using the **Right-Hand Rule** (definition follows).
The xy-plane, xz-plane, and yz-plane are called the **Coordinate Planes** determined by the axes, and these coordinate planes divide $\mathbb{R}^3$ into eight **Octants**.
If P is any point in $\mathbb{R}^3$, then we can represent P with an ordered triple (a, b, c) known as the **Coordinates** of P.

Definition

The **Right-Hand Rule** is a convention used to determine which axis is which, using your right hand. Form your right hand so that your thumb is sticking straight up (as if you are giving someone a "thumbs-up"), your pointer finger is pointing straight ahead (as if you are pointing at something), bend your middle finger 90° (as if your pointer and middle finger are "walking"), and curl your ring and pinky fingers towards you. In this orientation, your extended fingers and thumb represent axes.

Finger	**Axis**
Pointer	x-axis
Middle	y-axis
Thumb	z-axis

Technology Exploration: Use GeoGebra's 3D Graphing tool to explore the axes and the right-hand rule.

Example 1. By hand, draw a set of rectangular coordinate axes for $\mathbb{R}^3$, and label the positive axes. Then plot $P(1, 2, 3)$, $Q(2, -1, 5)$, and $R(4, 1, -2)$. Then, plot P, Q, and R in GeoGebra.

Exploration/Technology Exploration: Consider the following questions. Picture them in your mind, graph them by hand, and graph them in GeoGebra.

- What does $x = 2$ represent in $\mathbb{R}$? In $\mathbb{R}^2$? In $\mathbb{R}^3$?
- What does $x^2 + y^2 = 1$ represent in $\mathbb{R}^2$? In $\mathbb{R}^3$?
- What does $y = x^2$ represent in $\mathbb{R}^2$? In $\mathbb{R}^3$?
- What does $z = 1$ represent?

Exploration: Recall that in $\mathbb{R}^2$, the distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This formula is known as the distance formula. This formula is established by drawing a right triangle on the two points and using the Pythagorean Theorem.

Consider $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ as points in $\mathbb{R}^3$. Let's find the distance between P_1 and P_2 in a manner similar to that in $\mathbb{R}^2$.

Theorem

The Distance Formula
The distance $|P_1P_2|$ between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in $\mathbb{R}^3$ is given by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 2. Find the distance between $P(1, 2, 3)$ and $Q(2, -1, 5)$. Check your answer in GeoGebra.

Exploration: Let $P(x, y, z)$ be a point on a sphere of a sphere of radius r and center $C(h, k, \ell)$. Find an equation for the sphere.

Theorem

Every point $P(x, y, z)$ on a sphere of radius r and center $C(h, k, \ell)$ satisfies the equation

$$(x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2$$

Example 3. Find an equation of the unit sphere centered at the origin.

Example 4. Find an equation of a sphere whose center is $P(1, 2, 3)$ and that contains the point $Q(2, -1, 5)$. Graph the sphere in GeoGebra.

Example 5. Find the center and radius of the sphere $x^2 + y^2 - 4y + z^2 + 2z = 4$. Check your answer in GeoGebra.

2 Vectors

Definition

A **Vector** is a quantity that has both size (**Magnitude**) and direction. We often represent a vector as a directed line segment – an arrow with an **Initial Point** (or **Tail**) and a **Terminal Point** (or **Tip**). We write vectors with either bold font (whenever it is typed) or with an arrowhead (whenever it is handwritten).

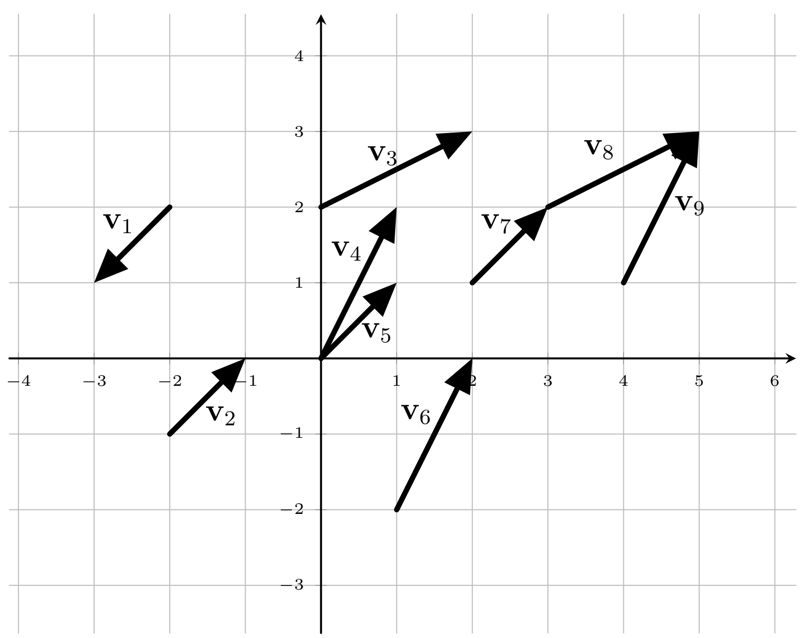
Exploration: Let's draw vectors $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{CD}$ for some points A, B, C, D.

Definition

Two vectors **u** and **v** are **Equivalent Vectors** if they have the same size and direction.

Note: Equivalent vectors do not need to have the same initial and terminal points.

Example 1. Below are nine vectors, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9$. Determine which vectors are equivalent.



Definition

The vector whose length is zero and has no direction is called the **Zero Vector**, denoted $\mathbf{0}$.

Definition

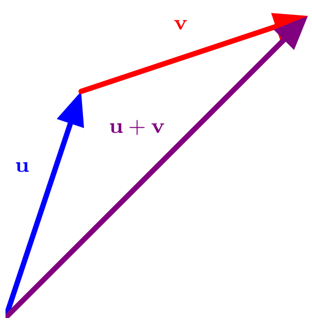
A vector whose initial point and terminal point represent linear motion is called a **Displacement Vector**.

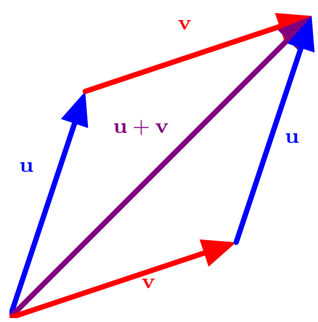
Definition

If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so that the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the **Sum of u and v** is the vector $\mathbf{u}+\mathbf{v}$ whose initial point is the initial point of $\mathbf{u}$ and whose terminal point is the terminal point of $\mathbf{v}$.

Note: We think of vector addition as the combination of displacements. More broadly, we have two addition laws called the Triangle Law of Addition and the Parallelogram Law of Addition that show graphical vector addition.

Triangle and Parallelogram Laws of Addition: Tail-to-Tip addition.





Definition

If c is a **Scalar** (i.e. a constant) and $\mathbf{v}$ is a vector, then the **Scalar Multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same if $c > 0$ and opposite if $c < 0$. If $c = 0$, then $c\mathbf{v} = \mathbf{0}$.
We call $-\mathbf{v} = (-1)\mathbf{v}$ the **Negative of v**.

Definition

Two vectors are **Parallel** if they are scalar multiples.

Note: Parallel vectors have the same direction; parallel vectors may differ in size.

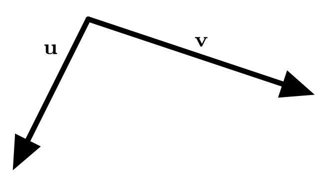
Definition

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, the **Difference of u and v** is the vector $\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v})$.

Note: In order to draw $\mathbf{u}-\mathbf{v}$, we first draw $-\mathbf{v}$, then we use the Triangle or Parallelogram Law of Addition.

Example 2. Below are two vectors $\mathbf{u}$ and $\mathbf{v}$. Draw and label the following vectors. You may wish to redraw $\mathbf{u}$ and $\mathbf{v}$ several times in order to draw these vectors.

a. $2\mathbf{u}$ b. $-3\mathbf{v}$ c. $\mathbf{u}+\mathbf{v}$ d. $\mathbf{v}-\mathbf{u}$ e. $\mathbf{u}-2\mathbf{v}$



Note: Everything that we have done so far has been with graphical definitions, but it has not involved algebra. If we happen to know locations and had some numerical values to represent the magnitude and direction of the vector, then we will see that it mostly works intuitively.

Definition

If $\mathbf{u}$ has its tail at the origin and its tip at the point (u_1, u_2) or (u_1, u_2, u_3), then the vector $\mathbf{u}$ is said to have **Coordinates** (u_1, u_2) or (u_1, u_2, u_3). We express the vector in **Component Form** as

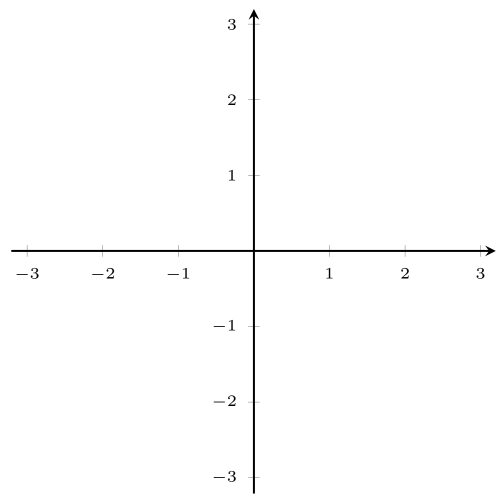
$$\mathbf{u} = \langle u_1, u_2 \rangle \quad \text{or} \quad \mathbf{u} = \langle u_1, u_2, u_3 \rangle$$

Definition

The graph of a vector of a fixed size and direction which has any particular initial point is called a **Representation** of a vector. The representation of a vector with initial point at the origin is called a **Position Vector**.
Given two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{u}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{u} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 3. Graph the points $A(3, 2)$ and $B(-2, -1)$, then graph the vector $\mathbf{u}$ whose representation is $\overrightarrow{AB}$. Write the component form of $\mathbf{u}$. Then use Geogebra to graph $\mathbf{u}$.



Definition

Given a vector $\mathbf{u}$, the **Magnitude** (or **Length** or **Norm**) of $\mathbf{u}$ is the length of any representation of $\mathbf{u}$, denoted with either $|\mathbf{u}|$ or $\|\mathbf{u}\|$.

Example 4. Find the length of $\mathbf{w} = \langle 1, 2 \rangle$.

Length of a Vector

If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2} \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Example 5. Find the length of $\mathbf{u} = \langle 1, 2, 3 \rangle$.

Definition

An n-dimensional vector is an ordered n-tuple $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$. Addition and scalar multiplication of n-dimensional vectors are defined just as with 2- and 3-dimensional vectors.

Component Vector Algebra

To add, subtract, and scale vectors given in component form, we add, subtract, and scale components, respectively. That is, if $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$, $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$, and $c \in \mathbb{R}$, then

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n \rangle \\ c\mathbf{u} &= \langle cu_1, cu_2, \ldots, cu_n \rangle \end{aligned}$$

Definition

We denote V_n to be the set of all n-dimensional vectors. That is, V_2 is the set of all 2-dimensional vectors, and V_3 is the set of all 3-dimensional vectors.

Theorem

Properties of Vectors:
If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V_n$ and $c, d \in \mathbb{R}$, then

i. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

ii. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

iii. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

iv. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

v. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

vi. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

vii. $(cd)\mathbf{u} = c(d\mathbf{u})$

viii. $1\mathbf{u} = \mathbf{u}$

Definition

A **Unit Vector** is a vector whose length is 1. If $\mathbf{u} \neq \mathbf{0}$, then the unit vector in the same direction as $\mathbf{u}$ is $\dfrac{1}{|\mathbf{u}|}\mathbf{u} = \dfrac{\mathbf{u}}{|\mathbf{u}|}$. The process of constructing the unit vector in the same direction as $\mathbf{u}$ is called **Normalizing u**.

Definition

The unit vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ in V_2 are called the **Standard Basis Vectors for** V_2. The unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ in V_3 are called the **Standard Basis Vectors for** V_3.

Example 6. Decompose $\mathbf{u} = \langle 1, 2, 3 \rangle$ into terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

Note: We can now generalize and conclude that

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Example 7. Find a unit vector in the opposite direction as $\mathbf{v} = \langle 2, -1, 5 \rangle$. Express this vector in terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

3 The Dot Product

Definition

The **Dot Product** (or **Inner Product** or **Scalar Product**) of two nonzero vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is the number

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example 1. Let $\mathbf{u} = \langle -3, 4, 2 \rangle$ and $\mathbf{v} = \langle 2, -1, 5 \rangle$. Find $\mathbf{u} \cdot \mathbf{v}$.

Example 2. Let $\mathbf{v} = \langle 2, -1, 5 \rangle$ and $\mathbf{w} = \langle -4, 2, -10 \rangle$. Find $\mathbf{v} \cdot \mathbf{w}$.

Properties of the Dot Product

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V_3$ and $c \in \mathbb{R}$, then

i. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \geq 0$

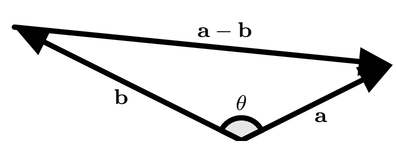
ii. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

iii. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

iv. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

v. $\mathbf{0} \cdot \mathbf{u} = 0$

Exploration: Consider the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Use the Law of Cosines to find a relationship between $\mathbf{u}$, $\mathbf{v}$, and θ.



Theorem

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, where $\theta \in [0, \pi]$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Corollary

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$.

Example 3. Let $\mathbf{v} = \langle 2, -1, 5 \rangle$ and $\mathbf{w} = \langle -1, 2, -3 \rangle$. Find the angle between $\mathbf{v}$ and $\mathbf{w}$. Round your conclusion to the nearest hundredth of a radian.

Example 4. Let $\mathbf{u} = \langle -3, 4, 2 \rangle$ and $\mathbf{v} = \langle 2, -1, 5 \rangle$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

Definition

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are called **Orthogonal** (or **Perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are called **Parallel** if $\mathbf{u} = c\mathbf{v}$ where $c \in \mathbb{R}$.

Exploration: Consider two vectors $\mathbf{u}$ and $\mathbf{v}$ with the same initial point P. Given these two vectors, we may want to find the projection of one onto the other.

Definition

Let $\mathbf{u}, \mathbf{v} \in V_n$. If we drop a perpendicular from $\mathbf{u}$ onto the line on $\mathbf{v}$ and call the point of intersection Q, then we can define two projections.

- The **Vector Projection of v onto u**, $\text{proj}_{\mathbf{u}} \mathbf{v}$, is the vector $\overrightarrow{PQ}$.
- The **Scalar Projection of v onto u**, $\text{comp}_{\mathbf{u}} \mathbf{v}$, is the signed magnitude of $\text{proj}_{\mathbf{u}} \mathbf{v}$, where the quantity is positive if $\text{proj}_{\mathbf{u}} \mathbf{v}$ is in the same direction as $\mathbf{u}$ and negative if in the opposite direction.

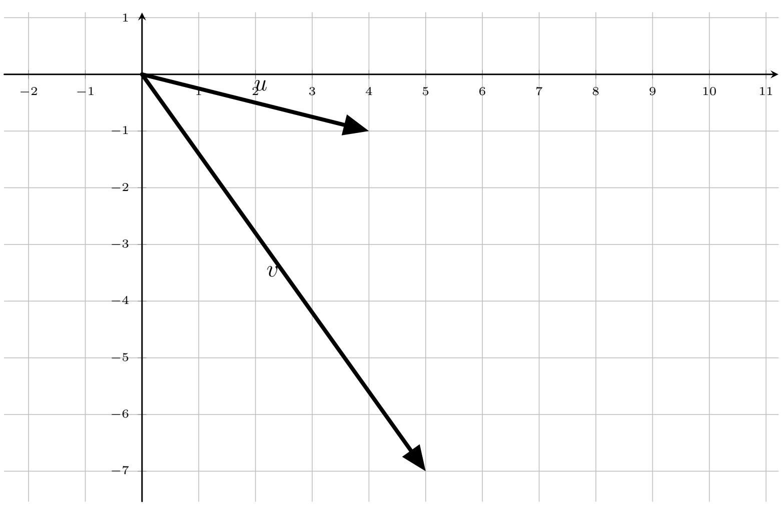
Exploration: Let's find nice formulas for $\text{comp}_{\mathbf{u}}\,\mathbf{v}$ and $\text{proj}_{\mathbf{u}}\,\mathbf{v}$ that involve $\mathbf{u}$ and $\mathbf{v}$.

Theorem

If $\mathbf{u}, \mathbf{v} \in V_n$, then

$$\begin{aligned}\text{comp}_{\mathbf{u}}\,\mathbf{v} &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}\\ \text{proj}_{\mathbf{u}}\,\mathbf{v} &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u}\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\end{aligned}$$

Example 5. Let $\mathbf{u} = \langle 4, -1\rangle$ and $\mathbf{v} = \langle 5, -7\rangle$.



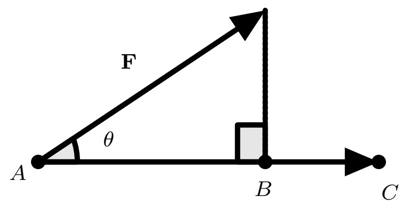
a. Find $\text{comp}_{\mathbf{u}}\,\mathbf{v}$ and $\text{proj}_{\mathbf{u}}\,\mathbf{v}$.

b. Find $\text{comp}_{\mathbf{v}}\,\mathbf{u}$ and $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

Exercise 1. Let $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle 2, -1, 5\rangle$. Find $\text{comp}_{\mathbf{u}}\,\mathbf{v}$ and $\text{proj}_{\mathbf{u}}\,\mathbf{v}$.

Exploration: Suppose a constant force is being exerted on an object to move it in a straight line. Then the force being exerted on the object has an *amount* of force as well as a *direction* of the force. So force can be represented by a vector! Suppose that the force, $\mathbf{F}$, is *not necessarily* being exerted in the same direction as the object is to move.

If the object moves from an initial point A to a terminal point C, then the displacement vector can be written as $\mathbf{D} = \overrightarrow{AC}$. The two vectors here are $\mathbf{D}$ and $\mathbf{F}$.



Notice that $\cos\theta = \dfrac{|\overrightarrow{AB}|}{|\mathbf{F}|}$, so $|\mathbf{F}|\cos\theta = |\overrightarrow{AB}|$. From physics, the work done by a vector $\mathbf{F}$ is defined as the product of the magnitude of the displacement, $|\mathbf{D}|$, and the magnitude of the applied force in the direction of the motion, $|\overrightarrow{AB}|$. In particular,

$$\begin{aligned} W &= |\mathbf{D}||\overrightarrow{AB}| \\ &= |\mathbf{D}|(|\mathbf{F}|\cos\theta) \\ &= |\mathbf{D}||\mathbf{F}|\cos\theta \\ &= \mathbf{D}\cdot\mathbf{F} \end{aligned}$$

Definition

The **Work** done by a force $\mathbf{F}$ on an object whose displacement vector is $\mathbf{D}$ is

$$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta$$

where θ is the angle between $\mathbf{F}$ and $\mathbf{D}$.

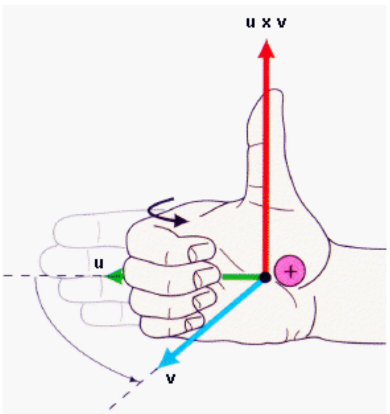
Units: Force is typically measured in pounds (lbs) (US) or Newtons (N) (metric).
Displacement is typically measured in feet (ft) (US) or meters (m) (metric).
Work is typically measured in foot-pounds (ft-lbs) (US) or Joules (J) (metric).

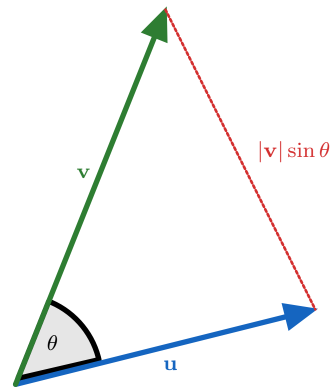
Example 6. A rolling backpack is being pulled a distance of 20 m along a horizontal path by a constant force of 50 N. The handle of the backpack is held at an angle of $55°$ above the horizontal. Find the work done by the force, rounded to the nearest Joule.

4 The Cross Product

Exploration: The **cross product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3 is another vector in V_3, called $\mathbf{u} \times \mathbf{v}$, that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. The length of this vector depends entirely upon both the lengths of $\mathbf{u}$ and $\mathbf{v}$ as well as the angle between them, θ.

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in V_3 with the same initial point, you may notice that there are two different directions that $\mathbf{u} \times \mathbf{v}$ could point. The appropriate direction is determined by the right-hand rule.





Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in V_3. Then the **Cross Product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} \times \mathbf{v}$ in V_3 such that

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin \theta \mathbf{n}$$

where $\mathbf{n}$ is the unit vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$ whose direction is determined by the right-hand rule. The vector $\mathbf{n}$ is called the **Normal Vector**.

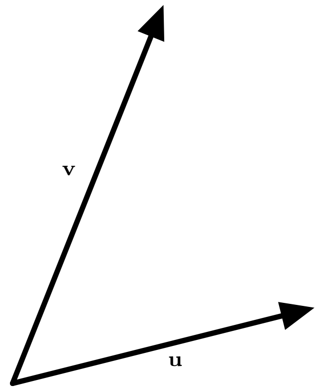
Note: The cross product is defined *only* for vectors in V_3 and produces another vector in V_3.

Theorem

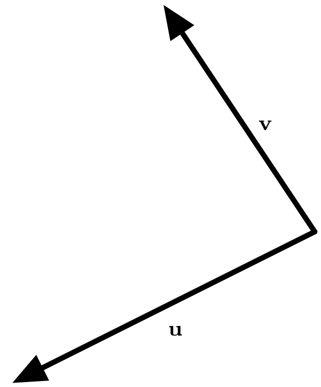
The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Example 1. Given the vectors $\mathbf{u}$ and $\mathbf{v}$, determine if $\mathbf{u} \times \mathbf{v}$ goes "out of the page" or "into the page".

a.



b.



Example 2. Let $\mathbf{u} = \langle 2, 4, 0 \rangle$ and $\mathbf{v} = \langle 1, 7, 0 \rangle$.

a. Find $\mathbf{u} \times \mathbf{v}$.

b. Use GeoGebra to graph $\mathbf{u}$ and $\mathbf{v}$.

c. Use GeoGebra to graph $\mathbf{u} \times \mathbf{v}$ using `Cross((2,4,0),(1,7,0))`. Visualize the vectors, the plane formed by the vectors, and the resulting cross product.

Example 3. Compute the following cross products, assuming they are V_3 vectors.

a. $\mathbf{i} \times \mathbf{j}$ b. $\mathbf{j} \times \mathbf{i}$ c. $\mathbf{i} \times \mathbf{k}$

Cross Product Properties

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V_3$ are nonzero and $c \in \mathbb{R}$, then

i. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

ii. $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v})$

iii. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

iv. $\mathbf{v} \times \mathbf{0} = \mathbf{0} \times \mathbf{v} = \mathbf{0}$

v. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

vi. $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

Definition

Component Definition of the Cross Product:
Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be two nonzero vectors in V_3. The **Cross Product** (or **Vector Product**) of two vectors $\mathbf{u}$ and $\mathbf{v}$, written $\mathbf{u}\times\mathbf{v}$, is the V_3 vector

$$\mathbf{u}\times\mathbf{v} = \langle u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1\rangle$$

Proof: We claim that the component definition of the cross product gives a vector orthogonal to

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be nonzero vectors in V_3. Write $\mathbf{c} = \langle c_1, c_2, c_3\rangle$.

If $\mathbf{u}$ and $\mathbf{c}$ are orthogonal, then $\mathbf{u}\cdot\mathbf{c} = 0$. If $\mathbf{v}$ and $\mathbf{c}$ are orthogonal, then $\mathbf{v}\cdot\mathbf{c} = 0$. Then

$$u_1c_1 + u_2c_2 + u_3c_3 = 0 \tag{1}$$

$$v_1c_1 + v_2c_2 + v_3c_3 = 0 \tag{2}$$

If we multiply (1) by v_3 and (2) by $-u_3$, then we get

$$u_1v_3c_1 + u_2v_3c_2 + u_3v_3c_3 = 0 \tag{3}$$

$$-u_3v_1c_1 - u_3v_2c_2 - u_3v_3c_3 = 0 \tag{4}$$

Adding (3) and (4), we get

$$(u_1v_3 - u_3v_1)c_1 + (u_2v_3 - u_3v_2)c_2 = 0 \tag{5}$$

If

$$c_1 = u_2v_3 - u_3v_2 \qquad \text{and} \qquad c_2 = -(u_1v_3 - u_3v_1)$$

then we have a legitimate solution. And if we substitute these into (1), we get

$$c_3 = u_1v_2 - u_2v_1$$

It follows that the resulting vector

$$\mathbf{c} = \langle u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1\rangle$$

is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. □

Example 4. Use the formula above to confirm $\mathbf{u}\times\mathbf{v}$ for the vectors $\mathbf{u} = \langle 2, 4, 0\rangle$ and $\mathbf{v} = \langle 1, 7, 0\rangle$.

Note: Though this formula is true, this is not an effective way to compute $\mathbf{u}\times\mathbf{v}$. Moreover, the original definition of the cross product requires us to know θ, and we may not have that information. The following information about matrices and determinants will cover topics from Linear Algebra very briefly and without justification. For more clarity, it is suggested to take Linear Algebra.

Definition

A **Matrix** is a rectangular array of numbers. For example, a 2×2 matrix looks like $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, and a 3×3 matrix looks like $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$.

Definition

The **Determinant** of a 2×2 matrix $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is given by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Example 5. Calculate the determinant $\begin{vmatrix} -6 & -4 \\ 1 & -3 \end{vmatrix}$. Confirm your result in GeoGebra using `Determinant({{-6,-4},{1,-3}})`.

Exercise 1. Calculate the determinant $\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$.

Definition

The **Determinant** of a 3×3 matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Note: Here is a neat trick for remembering this.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 6. Calculate the determinant $\begin{vmatrix} 2 & 3 & 1 \\ -3 & 6 & 4 \\ -5 & -1 & 3 \end{vmatrix}$. Confirm your result in GeoGebra.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are nonzero vectors in V_3, then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}$$

Note: This is a *complete* abuse of notation, just like thinking of $\frac{dy}{dx}$ as a fraction. However, this methodology will suffice for the purposes of computation, and it is common to do so.

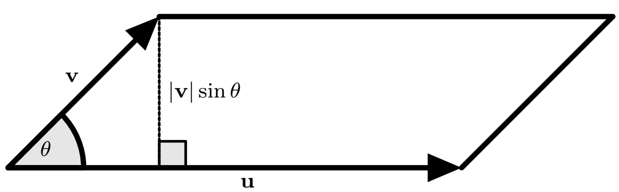
Example 7. Let $\mathbf{u} = \langle 1, 2, 3\rangle$ and $\mathbf{v} = \langle 2, -1, 5\rangle$. Find $\mathbf{u} \times \mathbf{v}$ by hand. Confirm your conclusion in GeoGebra.

Theorem

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Proof:

Exploration: The area of a parallelogram is given by base times height. So for the area of the parallelogram on two vectors $\mathbf{u}$ and $\mathbf{v}$ with the same initial point with θ being the smallest angle between them, we get $A = |\mathbf{u}||\mathbf{v}|\sin\theta$.



On the other hand, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin\theta||\mathbf{n}|$ Since $|\mathbf{n}| = 1$, and because $\sin\theta \geq 0$, we get

Theorem

The area of the parallelogram determined by the V_3 vectors $\mathbf{u}$ and $\mathbf{v}$ is $|\mathbf{u} \times \mathbf{v}|$.

Note: The area of the *triangle* determined by $\mathbf{u}$ and $\mathbf{v}$ would be $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$.

Example 8. Find the area of the parallelogram formed by $\langle 1, -2, 4\rangle$ and $\langle -3, 6, 7\rangle$.

5 Lines in 3D

Exploration: We will explore three different ways to describe a line in 3-space.

Suppose $Q(x_0, y_0, z_0)$ is a point in $\mathbb{R}^3$, $\mathbf{v} \in V_3$ is nonzero, and L is a line through P parallel to $\mathbf{v}$, where $P(x, y, z)$ is an arbitrary point on L.

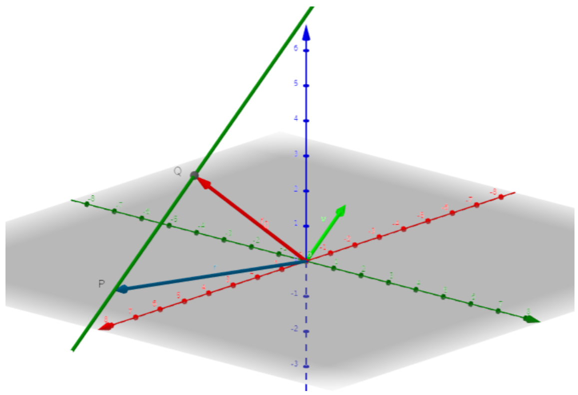


Figure 1: https://www.geogebra.org/classic/fsw99hgy

If $\mathbf{r} = \overrightarrow{OP}$ (the position vector of P), $\mathbf{r_0} = \overrightarrow{OQ}$ (position vector of Q), and $\mathbf{a} = \overrightarrow{QP}$, then the Triangle Law of Addition states

$$\mathbf{r} = \mathbf{r_0} + \mathbf{a}$$

Since $\mathbf{a}$ and $\mathbf{v}$ are parallel, $\mathbf{a} = t\mathbf{v}$ for some $t \in \mathbb{R}$. It follows that

Definition

The **Vector Equation** of a line L in $\mathbb{R}^3$ given by

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$$

where $\mathbf{r_0}$ is the position vector of a specific point on L, $\mathbf{v}$ is any vector parallel to L, and t is a **Parameter** which gives $\mathbf{r}$, the position vector of an arbitrary point on L.

Note: When $t > 0$, we get points on one side of Q on the line. When $t < 0$, we get points on the other side.

Exploration: If we write $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$, and $\mathbf{v} = \langle a, b, c \rangle$, then

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v} \qquad \Longleftrightarrow \qquad \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Definition

The equations

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

are the **Parametric Equations** of the line L through the point $Q(x_0, y_0, z_0)$ parallel to $\mathbf{v} = \langle a, b, c \rangle$, where $t \in \mathbb{R}$. Each value of t produces a point on L. The numbers a, b, and c are called the **Direction Numbers** of L.

Example 1. Let ℓ be the line parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ that passes through $P(1, 3, -4)$.

a. Find a vector equation for ℓ.

b. Find parametric equations for ℓ.

c. Find four points on ℓ.

d. Use GeoGebra to graph ℓ.

Definition

If the direction numbers for a line L are all nonzero, then the **Symmetric Equations** of L are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

If $a = 0$, then the symmetric equations of L are

$$x = x_0 \qquad , \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Similar symmetric equations hold when $b = 0$ and/or $c = 0$.

Note: Symmetric equations for a line are found by eliminating the parameter from parametric equations.

Example 2. Let ℓ be the line parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ that passes through $P(1, 3, -4)$. Find the symmetric equations for ℓ.

Example 3. Let ℓ be the line parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ that passes through $P(1, 3, -4)$. Find the points at which ℓ intersects each of the xy-, xz-, and yz-planes.

Definition

Two nonintersecting lines that are not parallel are called **Skew Lines**.

Example 4. Show that the lines ℓ_1 and ℓ_2 whose parametric equations are

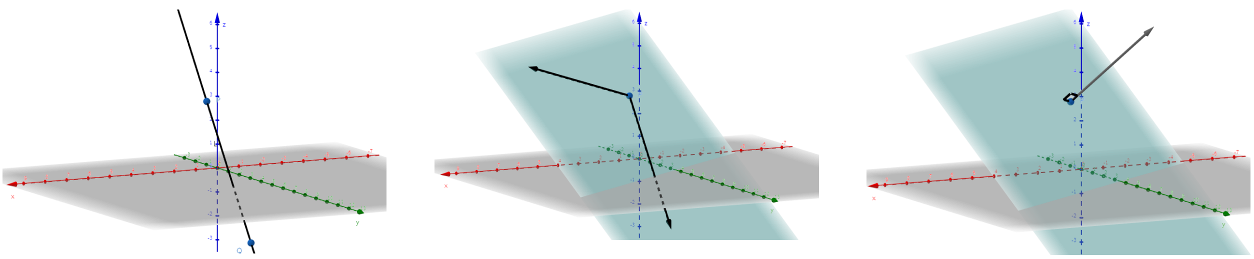
$$\begin{array}{llll} \ell_1 : & x = 1 + 2t & y = 3 - t & z = -4 + 5t \\ \ell_2 : & x = 1 + s & y = 2 - 3s & z = 3 + 2s \end{array}$$

are skew lines.

6 Planes

Exploration: We will explore three different ways to describe a plane in 3-space.

A line is determined either by two points or a point and a direction. A plane is determined by either three points or a point and two directions. However, instead of two separated directions, we can instead substitute those two directions with a single *orthogonal* direction.



Definition

A vector $\mathbf{n}$ orthogonal to a plane P is called a **Normal Vector** to P.

Definition

Let $Q(x_0, y_0, z_0)$ be a point on a plane in $\mathbb{R}^3$. If $P(x, y, z)$ is an arbitrary point on the plane, $\mathbf{r_0} = \langle x_0, y_0, z_0\rangle$ is the position vector for Q, and $\mathbf{r} = \langle x, y, z\rangle$ is the position vector for P, then a vector in the plane $\overrightarrow{QP}$ is given by

$$\mathbf{r} - \mathbf{r_0} = \langle x - x_0, y - y_0, z - z_0\rangle.$$

Moreover, if $\mathbf{n} = \langle a, b, c\rangle$ is a normal vector to the plane, then

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) = 0 \qquad \iff \mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r_0}$$

are each called a **Vector Equation of the Plane**.

Exploration: Since

$$\begin{aligned}\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) &= 0\\ \langle a, b, c\rangle\cdot\langle x - x_0, y - y_0, z - z_0\rangle &= 0\\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0\end{aligned}$$

Definition

The equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

is called the **Scalar Equation of the Plane Through** $Q(x_0, y_0, z_0)$ **with Normal Vector** $\langle a, b, c\rangle$.

Example 1. Find both the vector equation and the scalar equation of the plane through $P(1, 2, 3)$ with normal vector $\mathbf{n} = \langle 2, -1, 5 \rangle$. Find the intercepts of the plane, then sketch the plane.

Exploration: If we expand the scalar equation, we obtain a new form of an equation for the plane.

Definition

An equation of the form

$$ax + by + cz + d = 0$$

is called a **Linear Equation** in x, y, z. If a, b, c are not all 0, then the linear equation represents a plane with normal vector $\langle a, b, c \rangle$.

Example 2. Find a linear equation for the plane through $P(1, 2, 3)$ with normal vector $\mathbf{n} = \langle 2, -1, 5 \rangle$. Then graph the plane in GeoGebra.

Example 3. Find the point at which the line with parametric equations

$$x = 1 + 2t \quad , \quad y = 3 - t \quad , \quad z = -4 + 5t$$

intersects the plane $17x + 11y + 8z - 63 = 0$.

Definition

Two distinct planes in $\mathbb{R}^3$ are **Parallel** if the planes do not intersect. This occurs precisely when their normal vectors are parallel.

Theorem

Two planes in $\mathbb{R}^3$ that intersect are either coincident or intersect in a line. The angle between the planes is the same as the angle between their normal vectors.

Example 4. The two planes $3x - 14y - 19z + 14 = 0$ and $ax + by + cz - 4 = 0$ are parallel. Find a, b, c.

7 Functions and Surfaces

Definition

A **Function of Two Variables**, f, is a rule that assigns to each ordered pair (x, y) in a set D a unique number denoted $f(x, y)$. The set D is called the **Domain** of f, and its **Range** is the set of values that f attains: $\{f(x, y) \mid (x, y) \in D\}$. The variables x and y are called the **Independent Variables** of f.

Note: We will assume that the domain and range must contain only real numbers.

Note: Just like it is common to write $y = f(x)$, it is also common to write $z = f(x, y)$. In this case, the variable z is called the **Dependent Variable**. Notice D is a subset of $\mathbb{R}^2$, and the range is a subset of $\mathbb{R}$.

Example 1. Let P be the plane through the points $A(2, -1, 3)$, $B(1, -4, -2)$, and $C(0, 2, -1)$.

a. Find the linear equation for the plane P.

b. Write a formula for $z = f(x, y)$.

c. Graph $z = f(x, y)$ in GeoGebra.

d. Find $f(-1, 3)$.

e. What is the domain of f?

f. What is the range of f?

Example 2. Let $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$.

a. Find $f(4, 11)$.

b. Find and sketch the domain of f.

Definition

If f is a function of two variables with domain D, then the **Graph** of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$. The graph of a relationship of three variables is called a **Surface**.

Example 3. Use GeoGebra to graph $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$.

Example 4. Draw a set of coordinate axes for $\mathbb{R}^3$. Sketch the graph of $f(x, y) = 12 - 4x - 3y$ in $\mathbb{R}^3$ by hand on your axes.

Definition

A **Linear Function in Two Variables** is a function of the form $f(x, y) = ax+by+c$. The graph of a linear function in two variables is a plane.

Example 5. Draw a set of coordinate axes for $\mathbb{R}^3$. Sketch the graph of $f(x, y) = 4 - x^2$ in $\mathbb{R}^3$ by hand on your axes. Verify your results in GeoGebra.

Definition

The cross section of the graph of $f(x, y)$ intersecting with a plane is called a **Trace** of f in that plane.

Note: It is typically advantageous to graph a multivariable function by studying its traces in the planes $x = k$, $y = k$, and $z = k$, where $k \in \mathbb{R}$. The traces will often create a "wireframe" for your surface.

Example 6. Draw a set of coordinate axes for $\mathbb{R}^3$. Use traces to sketch the graph of $g(x, y) = 4x^2 + \frac{1}{9}y^2$ in $\mathbb{R}^3$ by hand on your axes. Verify your results in GeoGebra.

Example 7. Use GeoGebra to graph the following surfaces.

a. $f(x, y) = e^{\frac{x}{y}}$

b. $g(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

c. $h(x, y) = \dfrac{\sin x \sin y}{xy}$

d. $j(x, y) = \sin\left(xe^{\sin y}\right)$

Exploration: A linear equation is one of the form

$$ax + by + cz + d = 0.$$

The graph of such an equation is a plane.

A quadratic equation is one of the form

$$ax^2 + bxy + cy^2 + dyz + ez^2 + fxz + gx + hy + iz + j = 0.$$

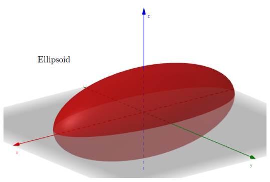
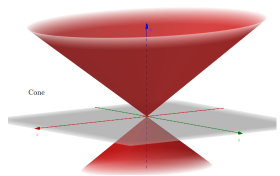
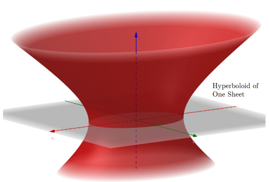
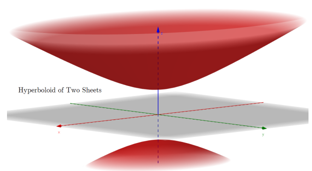
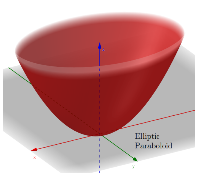
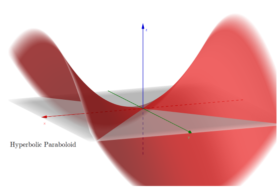
The graph of such an equation (if a, b, c, d, e, f are not all 0) is a **quadric surface**.

Definition

The graph of a quadratic equation in two variables is called a **Quadric Surface**.

Note: A quadric surface often does *not* correspond to an equation that represents a function.

Six basic quadric surfaces:

Surface	Equation	Surface	Equation
	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$		$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$
	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$		$\frac{-x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$		$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$

Example 8. Match each quadric surface with its appropriate equation.

a. $z = \dfrac{y^2}{4} - \dfrac{x^2}{3}$

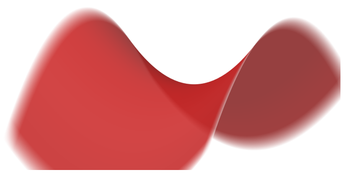
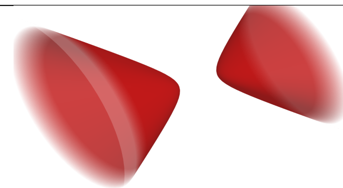
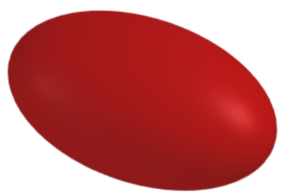
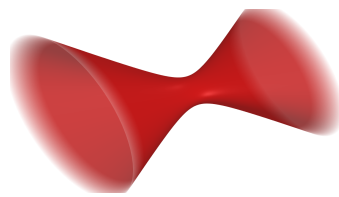
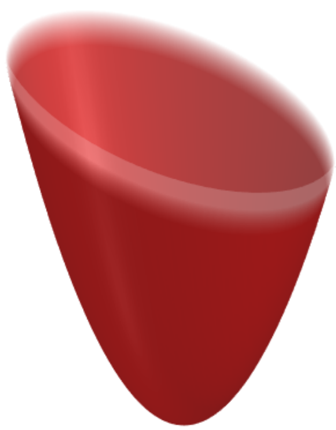
b. $\dfrac{x^2}{4} + z^2 - \dfrac{y^2}{4} = 1$

c. $\dfrac{z}{4} = \dfrac{x^2}{8} + \dfrac{y^2}{2}$

d. $-x^2 + y^2 - 4z^2 = 1$

e. $\dfrac{z^2}{4} = x^2 + \dfrac{y^2}{4}$

f. $\dfrac{x^2}{4} + y^2 + z^2 = 1$

___		___	
___		___	
___		___