MTH 254 Midterm Review Key

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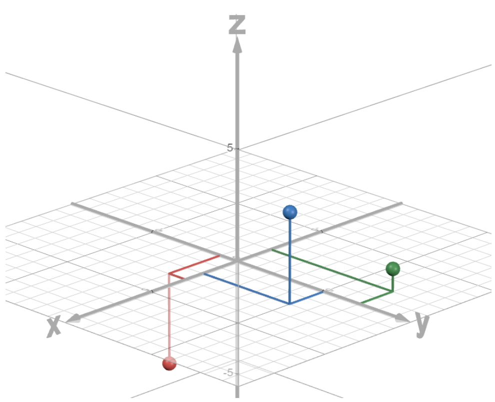
1. Let $P(3,-1,-4)$, $Q(2,5,4), R=(-2,7,1), \mathbf{u}=\overrightarrow{PQ}, \mathbf{v}=\overrightarrow{PR}$.
   (a) Draw a 3-dimensional rectangular coordinate system. Label the positive sides of the axes according to the right-hand rule, draw tick marks, and provide a scale.
   (b) Plot P, Q, and R. Be sure to include any guiding lines to give context to the points.
   (c) Graph $\mathbf{u}$ and $\mathbf{v}$.
   (d) Write $\mathbf{u}$ in component form.
   (e) Write $\mathbf{v}$ in terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.
   (f) Find $\mathbf{u}+\mathbf{v}$.
   (g) Find $\mathbf{u}-\mathbf{v}$.
   (h) Find $2\mathbf{u}+3\mathbf{v}$.

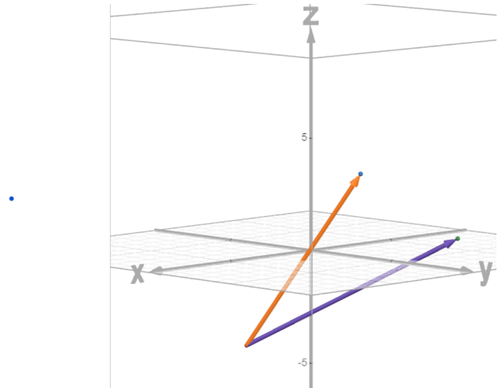
Solution:

(a) See graph below.



(b) See graph above.

(c) See graph below.



(d) $\mathbf{u} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle 2 - 3, 5 - (-1), 4 - (-4) \rangle = \langle -1, 6, 8 \rangle$

(e) $\mathbf{v} = \langle -2 - 3, 7 - (-1), 1 - (-4) \rangle = \langle -5, 8, 5 \rangle = -5\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$

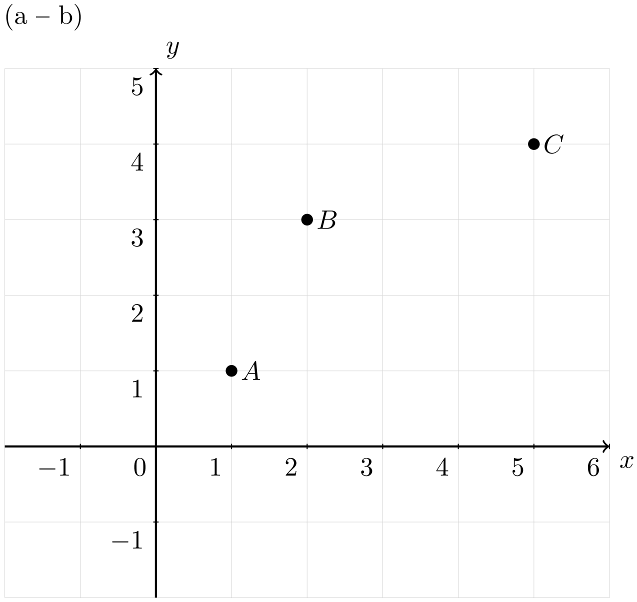
(f) $\mathbf{u} + \mathbf{v} = \langle -1, 6, 8 \rangle + \langle -5, 8, 5 \rangle = \langle -6, 14, 13 \rangle$

(g) $\mathbf{u} - \mathbf{v} = \langle -1, 6, 8 \rangle - \langle -5, 8, 5 \rangle = \langle 4, -2, 3 \rangle$

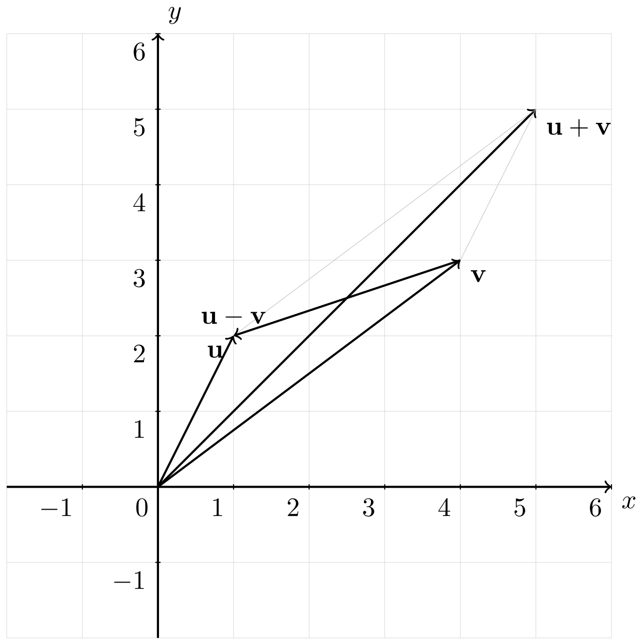
(h) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 6, 8 \rangle + 3\langle -5, 8, 5 \rangle = \langle -2, 12, 16 \rangle + \langle -15, 24, 15 \rangle = \langle -17, 36, 31 \rangle$

2. Let $A(1,1)$, $B(2,3)$, $C(5,4)$, $\mathbf{u} = \overrightarrow{AB}, \mathbf{v} = \overrightarrow{BC}$.
   (a) Draw a Cartesian plane. Label the positive sides of each axis, draw tick marks, and provide a scale.
   (b) Plot A, B, C.
   (c) Graph the position vectors for $\mathbf{u}$ and $\mathbf{v}$.
   (d) Graph $\mathbf{u} + \mathbf{v}$. Label the vector.
   (e) Graph $\mathbf{u} - \mathbf{v}$. Label the vector.
   (f) Graph $\text{proj}_{\mathbf{v}}\, \mathbf{u}$. Label the vector.
   (g) Graph $\text{proj}_{\mathbf{u}}\, \mathbf{v}$. Label the vector.

Solution:
(a – b)

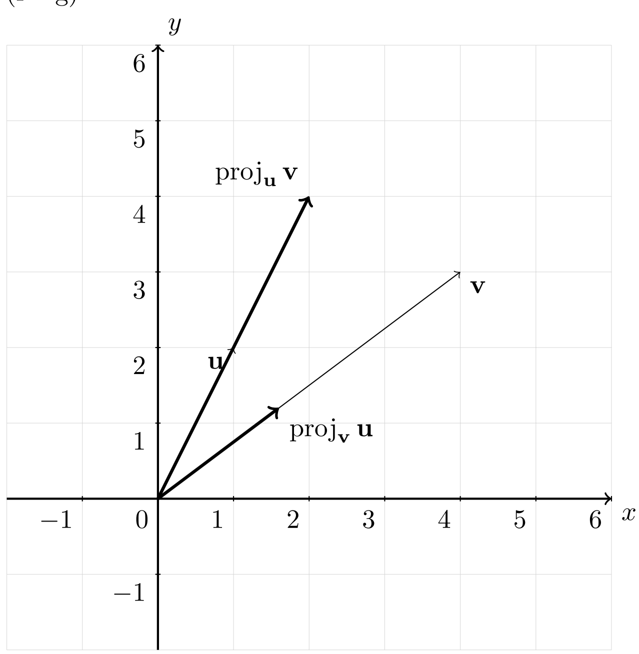


(c – e)



y
x
$\mathbf{u}+\mathbf{v}$
$\mathbf{v}$
$\mathbf{u}-\mathbf{v}$
$\mathbf{u}$
6
5
4
3
2
1
0
−1
−1
1
2
3
4
5
6

(f – g)



y
x
$\mathrm{proj}_{\mathbf{u}}\,\mathbf{v}$
$\mathbf{v}$
$\mathbf{u}$
$\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$
6
5
4
3
2
1
0
−1
−1
1
2
3
4
5
6

3. Let C be the curve determined by the vector function $\mathbf{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle$ with $2 \leq t \leq 4$. Find the exact length of C.

Solution: The length, L, of an arc is found by $L = \int_a^b |\mathbf{r}'(t)|\ dt$.

$$\begin{aligned} \mathbf{r}'(t) &= \langle 2, 2t, t^2 \rangle \\ |\mathbf{r}'(t)| &= \sqrt{(2)^2 + (2t)^2 + (t^2)^2} \\ &= \sqrt{4 + 4t^2 + t^4} \\ &= \sqrt{(2 + t^2)^2} \\ &= |2 + t^2| \end{aligned}$$

Because $2 + t^2 > 0$, $|\mathbf{r}'(t)| = 2 + t^2$. It follows that

$$\begin{aligned} L &= \int_a^b |\mathbf{r}'(t)|\ dt \\ &= \int_2^4 (2 + t^2)\ dt \\ &= \left[2t + \frac{1}{3}t^3 \right]_2^4 \\ &= \left(2(4) + \frac{1}{3}(4)^3 \right) - \left(2(2) + \frac{1}{3}(2)^3 \right) \\ &= 8 + \frac{64}{3} - 4 - \frac{8}{3} \\ &= \frac{68}{3} \end{aligned}$$

4. Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = \langle -3, 1, 4 \rangle$.

(a) Draw a 3-dimensional rectangular coordinate system. Label the positive sides of the axes according to the right-hand rule, draw tick marks, and provide a scale. Graph both $\mathbf{u}$ and $\mathbf{v}$ on your coordinate system.

(b) Find $|\mathbf{u}|$.

(c) Find a unit vector in the same direction as $\mathbf{u}$.

(d) Find the smallest angle between $\mathbf{u}$ and $\mathbf{v}$. Round your conclusion to the nearest tenth of a radian.

(e) Are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

(f) Find a nonzero vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

(g) Find the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

(h) Find $\text{proj}_{\mathbf{u}}\, \mathbf{v}$.

(i) Find $\text{comp}_{\mathbf{v}}\, \mathbf{u}$.

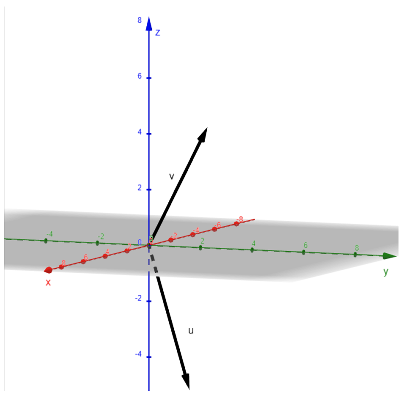
(j) Find the symmetric equations for the line passing through the terminal points of $\mathbf{u}$ and $\mathbf{v}$.

(k) Find the parametric equations for the line through the terminal point of the position vector for $\mathbf{u}$ with direction vector $\mathbf{v}$.

(l) Find a linear equation of the plane containing $\mathbf{u}, \mathbf{v}$, and $\mathbf{0}$.

Solution:

(a) See graph below.



(b) $|\mathbf{u}| = \sqrt{(1)^2 + (2)^2 + (-5)^2} = \sqrt{1 + 4 + 25} = \sqrt{30}$

(c) A unit vector in the same direction as $\mathbf{u}$ is $\frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{\sqrt{30}}(\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) = \frac{1}{\sqrt{30}}\mathbf{i} + \frac{2}{\sqrt{30}}\mathbf{j} - \frac{5}{\sqrt{30}}\mathbf{k}$

(d) Note that $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$, where θ is the smallest angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{b} &= (1)(-3) + (2)(1) + (-5)(4) = -21 \\
|\mathbf{v}| &= \sqrt{(-3)^2 + (1)^2 + (4)^2} = \sqrt{26} \\
\cos\theta &= \frac{-21}{\sqrt{30}\sqrt{26}} \\
\theta &= \arccos\left(\frac{-21}{\sqrt{780}}\right) \approx 2.42
\end{aligned}$$

(e) Since $\mathbf{u} \cdot \mathbf{v} = -21 \neq 0$, $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.

(f) A vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} \times \mathbf{v}$.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -5 \\ -3 & 1 & 4 \end{vmatrix} \\
&= 13\mathbf{i} + 11\mathbf{j} + 7\mathbf{k}
\end{aligned}$$

(g) The area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is $|\mathbf{u} \times \mathbf{v}| = \sqrt{13^2 + 11^2 + 7^2} = \sqrt{339}$.

(h) Recall $\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \dfrac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} = \frac{-21}{30}\langle 1, 2, -5\rangle = \left\langle \frac{-7}{10}, \frac{-7}{5}, \frac{7}{2} \right\rangle$.

(i) Recall $\operatorname{comp}_{\mathbf{v}} \mathbf{u} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{-21}{\sqrt{26}}$.

(j) The symmetric equations for a line are found by $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$. Since the line goes through the terminal points of $\mathbf{u}$ and $\mathbf{v}$, then the line goes through $(1, 2, -5)$ and has direction vector $\mathbf{u} - \mathbf{v} = \langle 1 - (-3), 2 - 1, -5 - 4\rangle = \langle 4, 1, -9\rangle$. Then the symmetric equations are

$$\frac{x - 1}{4} = \frac{y - 2}{1} = \frac{z + 5}{-9}$$

(k) The parametric equations are $x = x_0 + ta, y = y_0 + tb, z = z_0 + tc$. This is the same point as in the last prompt, but the direction numbers are $\mathbf{v} = \langle -3, 1, 4\rangle$, so

$$\begin{aligned} x &= 1 - 3t \\ y &= 2 + t \\ z &= -5 + 4t \end{aligned}$$

(l) The linear equation for the plane containing $\mathbf{u}, \mathbf{v}, \mathbf{0}$ is $\alpha x + \beta y + \gamma z + d = 0$, where $\langle \alpha, \beta, \gamma\rangle$ is a vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$, and $d \in \mathbb{R}$. Since the plane goes through $(0, 0, 0)$, $d = 0$. Moreover, $\langle \alpha, \beta, \gamma\rangle = \mathbf{u} \times \mathbf{v}$ will work. Then the equation is $13x + 11y + 7z = 0$.

5. Let $\mathbf{r}(t) = \langle t^3 - 4t, \sin\left(\frac{\pi}{3}t\right)\rangle$ with $-3 \le t \le 3$. Let C be the curve determined by $\mathbf{r}$.

(a) Produce a table of values to find the points on C.

(b) Draw a Cartesian plane, and sketch C. Include arrows to indicate the direction that $\mathbf{r}$ travels as t increases.

(c) Graph $\mathbf{r}(1)$. Label the vector.

(d) Graph $\mathbf{T}(1)$. Label the vector.

(e) Graph $\mathbf{N}(1)$. Label the vector.

(f) Find $\mathbf{T}(1)$.

(g) Find an equation of the tangent line to C at the point where $t = 1$.

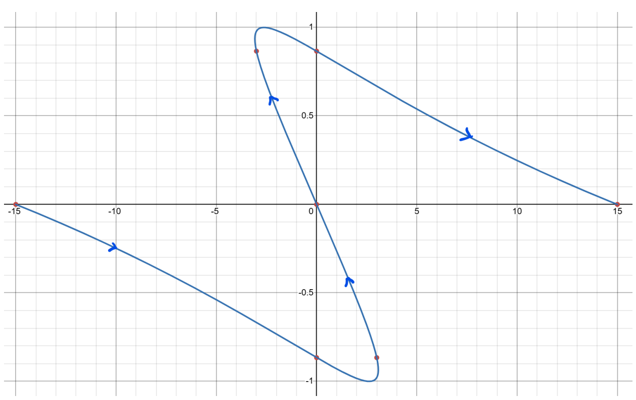
(h) Find the curvature of C when $t = 1$. Round your conclusion to the nearest hundredth.

Solution:

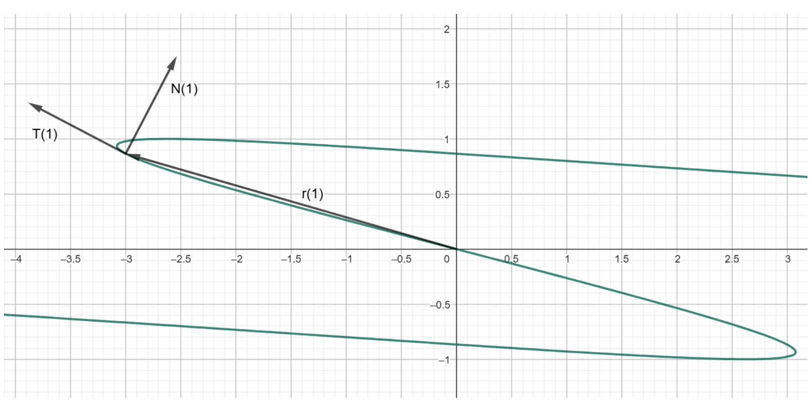
(a) We will make a table of t-, x-, and y-values.

t	$x = t^3 - 4t$	$y = \sin\left(\frac{\pi}{3}t\right)$
-3	-15	0
-2	0	$\frac{-\sqrt{3}}{2}$
-1	3	$\frac{-\sqrt{3}}{2}$
0	0	0
1	-3	$\frac{\sqrt{3}}{2}$
2	0	$\frac{\sqrt{3}}{2}$
3	15	0

(b) See graph below.



(c) See graph below. Notice the scale has changed from the last part.



(d) See above.

(e) See above.

(f) Recall $\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|}$. Now, $\mathbf{r}'(t) = \left\langle 3t^2 - 4, \frac{\pi}{3}\cos\left(\frac{\pi}{3}t\right)\right\rangle$. Then $\mathbf{r}'(1) = \left\langle -1, \frac{\pi}{6}\right\rangle$, and $|\mathbf{r}'(1)| = \sqrt{1 + \frac{\pi^2}{36}}$. It follows that $\mathbf{T}(1) = \left\langle \frac{-1}{\sqrt{1+\frac{\pi^2}{36}}}, \frac{\pi}{6\sqrt{1+\frac{\pi^2}{36}}}\right\rangle$.

(g) Since $\mathbf{r}(1) = \left\langle -3, \frac{\sqrt{3}}{2}\right\rangle$, the tangent line will pass through $\left(-3, \frac{\sqrt{3}}{2}\right)$. Since $\mathbf{r}'(1) = \left\langle -1, \frac{\pi}{6}\right\rangle$, the slope of the tangent line to C at $\left(-3, \frac{\sqrt{3}}{2}\right)$ is $\dfrac{\frac{\pi}{6}}{-1} = \frac{-\pi}{6}$. It follows that an equation to the tangent line is $y - \frac{\sqrt{3}}{2} = \frac{-\pi}{6}(x+3)$.

(h) There are several equations for the curvature of C when $t = 1$, but one is $\dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3}$. This only works in three-dimensional space, so we will consider $\mathbf{r}$ to have a third component of 0. Then

$$\begin{aligned}
\mathbf{r}'(1) &= \left\langle -1, \frac{\pi}{6}, 0\right\rangle \\
|\mathbf{r}'(1)| &= \sqrt{1 + \frac{\pi^2}{36}} \\
\mathbf{r}''(t) &= \left\langle 6t, \frac{-\pi^2}{9}\sin\left(\frac{\pi}{3}t\right), 0\right\rangle \\
\mathbf{r}''(1) &= \left\langle 6, \frac{-\pi^2\sqrt{3}}{18}, 0\right\rangle \\
\mathbf{r}'(1) \times \mathbf{r}''(1) &= \left\langle 0, 0, \frac{-\pi^2\sqrt{3}}{18} - \pi\right\rangle \\
|\mathbf{r}'(1) \times \mathbf{r}''(1)| &= \frac{\pi^2\sqrt{3}}{18} + \pi \\
\kappa(1) &= \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} \\
&= \frac{\frac{\pi^2\sqrt{3}}{18} + \pi}{\sqrt{1 + \frac{\pi^2}{36}}^{\,3}} \\
&\approx 2.84
\end{aligned}$$

6. Suppose a particle is moving in space with initial position $\mathbf{r}(0) = \mathbf{i} + \mathbf{k}$ and velocity

$$\mathbf{v}(t) = \left\langle \frac{2}{1+t^2}, 5e^{5t-5}, \frac{4}{t+1} \right\rangle$$

where $\mathbf{v}(t)$ is measured in meters per second. Let C represent the path the particle takes as t increases. Find the following quantities. Use exact values, and respond to each part with a sentence with units.

(a) The velocity of the particle after 1 second.

(b) The position of the particle at time t.

(c) The position of the particle after 1 second.

(d) The acceleration of the particle at time t.

(e) The acceleration of the particle after 1 second.

(f) The exact displacement vector of the particle in the first 3 seconds.

(g) The tangential component of acceleration after 1 second.

(h) The normal component of acceleration after 1 second.

(i) The curvature of C at the point when $t = 1$.

Solution:

(a) The velocity is found by evaluating $\mathbf{v}$ at $t = 1$. So

$$\mathbf{v}(1) = \left\langle \frac{2}{1+(1)^2}, 5e^{5(1)-5}, \frac{4}{(1)+1} \right\rangle = \langle 1, 5, 2 \rangle$$

(b) The position is found by integrating the velocity function.

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ dt = \int \left\langle \frac{2}{1+t^2}, 5e^{5t-5}, \frac{4}{t+1} \right\rangle\ dt = \left\langle 2\arctan t, e^{5t-5}, 4\ln|t+1| \right\rangle + \langle c_1, c_2, c_3 \rangle$$

Since $\mathbf{r}(0) = \mathbf{i} + \mathbf{k}$,

$$\begin{aligned}
\langle 1, 0, 1 \rangle &= \left\langle 2\arctan(0), e^{5(0)-5}, 4\ln|(0)+1| \right\rangle + \langle c_1, c_2, c_3 \rangle \\
\langle 1, 0, 1 \rangle &= \langle 0, e^{-5}, 0 \rangle + \langle c_1, c_2, c_3 \rangle \\
\langle 1, 0, 1 \rangle &= \langle c_1, e^{-5} + c_2, c_3 \rangle
\end{aligned}$$

Thus, $\langle c_1, c_2, c_3 \rangle = \langle 1, -e^{-5}, 1 \rangle$. Then

$$\mathbf{r}(t) = \left\langle 2\arctan(t) + 1, e^{5t-5} - e^{-5}, 4\ln|t+1| + 1 \right\rangle$$

(c) The position at $t = 1$ is $\mathbf{r}(1)$, measured in meters.

$$\begin{aligned}
\mathbf{r}(1) &= \left\langle 2\arctan(1) + 1, e^{5(1)-5} - e^{-5}, 4\ln|1+1| + 1 \right\rangle \\
&= \left\langle \frac{\pi}{2} + 1, 1 - e^{-5}, 4\ln(2) + 1 \right\rangle
\end{aligned}$$

(d) The acceleration is found by $\mathbf{a}(t) = \mathbf{v}'(t)$.

$$\begin{aligned}
\mathbf{a}(t) &= \frac{d}{dt}(\mathbf{v}(t)) \\
&= \left\langle \frac{-4t}{(1+t^2)^2}, 25e^{5t-5}, \frac{-4}{(t+1)^2} \right\rangle
\end{aligned}$$

(e) The acceleration at time $t=1$ is $\mathbf{a}(1)$, measured in $\frac{\text{m}}{\text{s}^2}$.

$$\begin{aligned}\mathbf{a}(1) &= \left\langle \frac{-4(1)}{(1+(1)^2)^2}, 25e^{5(1)-5}, \frac{-4}{((1)+1)^2} \right\rangle \\ &= \langle -1, 25, -1 \rangle\end{aligned}$$

(f) The displacement vector in the first three seconds is found by $\mathbf{r}(3) - \mathbf{r}(0)$.

$$\begin{aligned}\mathbf{r}(3) - \mathbf{r}(0) &= \left\langle 2\arctan(3) + 1, e^{5(3)-5} - e^{-5}, 4\ln|(3)+1| + 1 \right\rangle - (\mathbf{i} + \mathbf{k}) \\ &= \left\langle 2\arctan(3) + 1, e^{10} - e^{-5}, 4\ln(4) + 1 \right\rangle - \langle 1, 0, 1 \rangle \\ &= \left\langle 2\arctan(3), e^{10} - e^{-5}, 4\ln 4 \right\rangle\end{aligned}$$

(g) The tangential component of acceleration after 1 second is $a_T = \dfrac{\mathbf{r}'(1) \cdot \mathbf{r}''(1)}{|\mathbf{r}'(1)|} = \dfrac{\mathbf{v}(1) \cdot \mathbf{a}(1)}{|\mathbf{v}(1)|}$. It follows that

$$\begin{aligned}\mathbf{v}(1) \cdot \mathbf{a}(1) &= \langle 1, 5, 2 \rangle \cdot \langle -1, 25, -1 \rangle \\ &= -1 + 125 - 2 \\ &= 122 \\ |\mathbf{v}(1)| &= \sqrt{(1)^2 + (5)^2 + (2)^2} \\ &= \sqrt{30} \\ a_T &= \frac{122}{\sqrt{30}}\end{aligned}$$

(h) The normal component of acceleration after 1 second is $a_N = \dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|} = \dfrac{|\mathbf{v}(1) \times \mathbf{a}(1)|}{|\mathbf{v}(1)|}$. It follows that

$$\begin{aligned}\mathbf{v}(1) \times \mathbf{a}(1) &= \langle 1, 5, 2 \rangle \times \langle -1, 25, -1 \rangle \\ &= \langle -55, -1, 30 \rangle \\ |\mathbf{v}(1) \times \mathbf{a}(1)| &= \sqrt{(-55)^2 + (-1)^2 + (30)^2} \\ &= \sqrt{3926} \\ a_N &= \frac{\sqrt{3926}}{\sqrt{30}} \\ &= \sqrt{\frac{1963}{15}}\end{aligned}$$

(i) The curvature at time $t=1$ is $\dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{|\mathbf{v}(1) \times \mathbf{a}(1)|}{|\mathbf{v}(1)|^3}$. It follows that

$$\begin{aligned}\kappa(1) &= \frac{\sqrt{3926}}{\sqrt{30}^3} \\ &= \frac{\sqrt{1963}}{30\sqrt{15}}\end{aligned}$$