MTH 261 Final Exam Sample

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Winter 2020

Name: ______________________________

For instructor use only

Score : /125

Some notes from Damien about the exam:

- You will have 2 hours 20 minutes to complete this exam. When you are finished, please turn it in to Damien.
- You may use a graphing calculator, but once you begin the exam, you *may not* use an internet sources or any other external source for assistance. This includes your notes, peers, textbook, tutors, other human beings, robots, or any other source I may have failed to mention.
- You may ask Damien for clarification on any problems, but you *may not* converse with anyone else regarding this exam.
- If you have any questions, ask Damien!
- The use of scratch paper is encouraged and should be included at the end of the exam. Please clearly label all work for each prompt.
- Please read all of the instructions carefully.
- Please show all of your work to each prompt. This is exceedingly important!
- Please use exact values unless otherwise stated.
- Please use vector notation any time you are writing a vector.
- Please show all row reduction steps.
- May the Force be with you!

Furthermore, please read all of the conditions below before beginning the exam. Indicate that you have read and understand these conditions by placing your initials on the line below.

- I have and will follow all of the guidelines stated above.
- I have not and will not cheat or violate any part of the Academic Integrity Policies.
- I will not provide or distribute any portion of this document to anyone other than the instructor.
- All work attached herein is my own authentic work.

Initials : ______________

(5) 1. Let $A = \begin{bmatrix} \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ A & K & Q & J \\ \flat & \sharp & \natural & ♩ \end{bmatrix}$. Suppose $\det A = 48$.

(a) Find $\det \begin{bmatrix} \flat & \sharp & \natural & ♩ \\ \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ A & K & Q & J \end{bmatrix}$.

Solution: Since this matrix is obtained from A by two interchanges, the determinant is 48.

(b) Find $\det \begin{bmatrix} -2\clubsuit & -2\diamondsuit & -2\heartsuit & -2\spadesuit \\ 0.1A & 0.1K & 0.1Q & 0.1J \\ \flat + 2A & \sharp + 2K & \natural + 2Q & ♩ + 2J \end{bmatrix}$.

Solution: This matrix is obtained from A by scaling the first row by -2, scaling the second row by 0.1, and a replacement in row 3. Replacement does not affect the determinant, but the scalings each scale the determinant. This means that the determinant is $48(-2)(0.1) = -9.6$.

(9) 2. Let $A \in M_{16\times 23}$ such that $\operatorname{rank} A = 15$. Find the following. *Show any arithmetic computations for partial credit consideration.*

(a) $\dim \operatorname{Row} A =$ 15

(b) $\dim \operatorname{Nul} A =$ 8

(c) $\dim \operatorname{LNul} A =$ 1

(9) 3. A vector space is a nonempty set V of vectors on which two operations are defined – addition (+) and scalar multiplication ($\times$) – via ten axioms that must be true for all $\mathbf{u}, \mathbf{v} \in V$ and $c, d \in \mathbb{R}$. Which of the following are vector space **axioms**? (Circle all that apply)

(a) $1 \times \mathbf{v} = \mathbf{v}$

(b) $\boxed{0 \times \mathbf{v} = \mathbf{0}}$

(c) $c + \mathbf{v} \in V$

(d) $\boxed{(cd) \times \mathbf{v} = c \times (d \times \mathbf{v})}$

(e) $\mathbf{u} \times \mathbf{v} \in V$

(f) $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent

(g) $\boxed{\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}}$

(h) $\boxed{c \times (\mathbf{u} + \mathbf{v}) = (c \times \mathbf{u}) + (c \times \mathbf{v})}$

(i) $(-1) \times \mathbf{u} = -\mathbf{u}$

(10) 4. Let $A \in M_{n\times n}$.

The Invertible Matrix Theorem is a massive theorem which states that a series of statements are equivalent. On the other hand, the theorem implies a Singular Matrix Theorem. Which of the following are equivalent to the statement A **is an invertible matrix.**? (Circle all that apply)

(a) $\operatorname{Nul} A$ consists of more than one vector

(b) $\operatorname{rank} A < n$

(c) $\boxed{\text{The columns of } A \text{ are linearly independent}}$

(d) The equation $A\mathbf{x} = \mathbf{0}$ is inconsistent

(e) 0 is an eigenvalue of A

(f) $\boxed{\text{The linear transformation } \mathbf{x} \mapsto A\mathbf{x} \text{ is onto.}}$

(g) A is not square

(h) $\det A = 0$

(i) $\boxed{A^T \text{ is invertible}}$

(j) $\boxed{A \text{ is row equivalent to } I}$

(5) 5. Explain as specifically as possible what it means for a set $\mathcal{B}$ to be a **basis** for a vector space V.

Solution: The set $\mathcal{B}$ is a basis for a vector space V if $\mathcal{B}$ is linearly independent and spans V.

6. Find the following determinants.

(3) (a) $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

Solution: $\det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = (1)(3) - (1)(2) = 1$

(2) (b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 3 & 5 & 8 & 0 \\ 13 & 21 & 34 & 55 \end{vmatrix}$

Solution: This matrix is triangular, so its determinant is the product of the diagonal entries. Therefore, $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 3 & 5 & 8 & 0 \\ 13 & 21 & 34 & 55 \end{vmatrix} = (1)(2)(8)(55) = 880.$

(4) (c) $\begin{vmatrix} -1 & 2 & -3 \\ 3 & -2 & 1 \\ 0 & -2 & 0 \end{vmatrix}$

Solution:

$$\begin{aligned}
\begin{vmatrix} -1 & 2 & -3 \\ 3 & -2 & 1 \\ 0 & -2 & 0 \end{vmatrix} &= (-1) \begin{vmatrix} 1 & -2 & 3 \\ 3 & -2 & 1 \\ 0 & -2 & 0 \end{vmatrix} \\
&= (-1) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 4 & -8 \\ 0 & -2 & 0 \end{vmatrix} \\
&= (-1)(4) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 0 \end{vmatrix} \\
&= (-1)(4) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -4 \end{vmatrix} \\
&= (-1)(4)(-4) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \\
&= (-1)(4)(-4)(1)(1)(1) \\
&= 16
\end{aligned}$$

(10) 7. The set $GL_2(\mathbb{R})$ is the set of all invertible 2×2 matrices. Determine if $GL_2(\mathbb{R})$ is a subspace of $M_{2\times 2}$ or not. Show all work to support your conclusion.

> **Solution:** The set $GL_2(\mathbb{R})$ is a subspace of $M_{2\times 2}$ if it is nonempty, closed under addition, and closed under scalar multiplication.
>
> Suppose $A \in GL_2(\mathbb{R})$. Note that $0A = 0$, but 0 is not an invertible 2×2 matrix. Therefore, $GL_2(\mathbb{R})$ is not closed under scalar multiplication. It follows that $GL_2(\mathbb{R})$ is not a subspace of $M_{2\times 2}$.

8. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \in \mathbb{R}^3$, and let $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix}$. Suppose RREF $A = \begin{bmatrix} 1 & 0 & 4 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(3) (a) Does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ span $\mathbb{R}^3$? Briefly justify your answer.

> **Solution:** Three pivot columns are required for $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ to span $\mathbb{R}^3$. There are only two pivot columns, so the answer is no.

(3) (b) Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ linearly independent? If yes, briefly justify your answer. If no, **provide a linear dependence relation** among $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and $\mathbf{u}_4$.

> **Solution:** The set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent, because there are more vectors than entries in those vectors.
>
> Consider the homogeneous equation $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + x_4\mathbf{u}_4 = \mathbf{0}$.
>
> Note that RREF $\begin{bmatrix} A & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 & 4 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. This tells us that $x_1 + 4x_3 + 4x_4 = 0$, and $x_2 - x_3 + x_4 = 0$, with x_3, x_4 being free. If we allow $x_3 = 1$ and $x_4 = 0$, then we get $(x_1, x_2, x_3, x_4) = (-4, 1, 1, 0)$. Substituting this into our homogeneous equation, we get our linear dependence relation below.
>
> $$-4\mathbf{u}_1 + 1\mathbf{u}_2 + 1\mathbf{u}_3 = \mathbf{0}$$

(2) (c) Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ a basis for $\mathbb{R}^4$? Why or why not?

> **Solution:** No, this set is linearly dependent, so it cannot be a basis.

9. Let H be the subspace of $\mathbb{R}^3$ spanned by $S = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\3\\5\end{bmatrix} \right\}$.

(2) (a) Is S linearly independent or linearly dependent?

Solution: S is linearly dependent because there are more vectors than entries in those vectors.

(3) (b) Find a basis for H consisting only of vectors in S.

Solution: Notice that $\begin{bmatrix}2\\2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix}$, so

$$\operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\3\\5\end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\3\\5\end{bmatrix} \right\}$$

Moreover, $\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}$, so

$$\operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\3\\5\end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\3\\5\end{bmatrix} \right\}$$

Lastly, $\begin{bmatrix}3\\3\\5\end{bmatrix} = 3\begin{bmatrix}1\\1\\1\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix}$, so

$$\operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\3\\5\end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$

Since $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ are not scalar multiples of each other, $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ is a basis for H.

(2) (c) Find $\dim H$.

Solution: Since a basis for H has two vector, $\dim H = 2$.

(3) (d) Expand the basis you found in part (b) to a basis for $\mathbb{R}^3$.

Solution: A basis for $\mathbb{R}^3$ can be found by including an additional vector to $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ that is not in H. Thus, $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

10. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & -1 & 2 \\ 2 & 4 & 2 & 0 \end{bmatrix}$.

(5) (a) Find a basis for $\operatorname{Col} A$.

Solution: A basis for $\operatorname{Col} A$ is the set of pivot columns of A. Let's find RREF A.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & -1 & 2 \\ 2 & 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 2 & 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and third columns of A are pivot columns, so a basis for $\operatorname{Col} A$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$.

(5) (b) Find a basis for $\operatorname{Nul} A$.

Solution: Consider the homogeneous equation $A\mathbf{x} = \mathbf{0}$. We can solve this by row-reducing $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$. From the previous part,

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this, we obtain that $x_1 + 2x_2 + x_4 = 0$ and $x_3 - x_4 = 0$. Thus

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

It follows that a basis for $\operatorname{Nul} A$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(2) (c) $\operatorname{rank} A = \underline{\quad 2 \quad}$

(2) (d) $\operatorname{nullity} A = \underline{\quad 2 \quad}$

(2) (e) $\operatorname{Col} A$ is a subspace of $\mathbb{R}^m$ for $m = \underline{\quad 3 \quad}$.

(2) (f) $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$ for $n = \underline{\quad 4 \quad}$.

11. Consider the polynomials $\mathbf{p}_1(t) = 1 + t + 2t^2$, $\mathbf{p}_2(t) = -t + 2t^2$, $\mathbf{p}_3(t) = 1 + 2t$, and $\mathbf{p}_4(t) = 1 + t + t^2$. Let $P = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$.

(3) (a) Does P span $\mathbb{P}_2$? Briefly justify your answer.

Solution: To determine span P, we can find the coordinate vectors for each vector in P. If $S = \{1, t, t^2\}$ is the standard basis for $\mathbb{P}_2$, then

$$[\mathbf{p}_1]_S = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad , \quad [\mathbf{p}_2]_S = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad , \quad [\mathbf{p}_3]_S = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad , \quad [\mathbf{p}_4]_S = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let's augment these vectors together and row-reduce to determine the span.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 2 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since we have 3 pivot columns, P spans a 3-dimensional space. Since $\dim \mathbb{P}_2 = 3$, P spans $\mathbb{P}_2$.

(3) (b) Is P linearly independent? If not, find a dependence relation for the polynomials.

Solution: Based on the last part, not every column of our matrix was a pivot column, so P is linearly dependent.
Consider the homogeneous equation $x_1[\mathbf{p}_1]_S + x_2[\mathbf{p}_2]_S + x_3[\mathbf{p}_3]_S + x_4[\mathbf{p}_4]_S = \mathbf{0}$. From the last part, we know that $\begin{bmatrix}[\mathbf{p}_1]_S & \mathbf{p}_2]_S & \mathbf{p}_3]_S & \mathbf{p}_4]_S & \mathbf{0}\end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. So then $x_1 + x_3 = 0$, $x_2 - x_3 = 0$, and $x_4 = 0$. If $x_3 = 1$, then $x_1 = -1$, $x_2 = 1$, $x_3 = 1$, and $x_4 = 0$. Thus, we have the relationship that

$$-\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}$$

(3) (c) If possible, find a basis for $\mathbb{P}_2$ consisting only of the vectors from $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$. If this is not possible, briefly state why.

Solution: Because $\begin{bmatrix}[\mathbf{p}_1]_S & \mathbf{p}_2]_S & \mathbf{p}_3]_S & \mathbf{p}_4]_S\end{bmatrix}$ has pivot columns in the first, second, and fourth columns, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$ is a basis for $\mathbb{P}_2$.

(5) 12. Let $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = 2 + t$, $\mathbf{v}_3 = 3 + 2t + t^2$, and $v_4 = 4 + 3t + 2t^2 + t^3$. Then $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{P}_3$. If $\left[\mathbf{x}\right]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$, then find $\mathbf{x}$.

Solution: By definition, $\mathbf{x} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{bmatrix} [\mathbf{x}]_{\mathcal{B}}$. Therefore,

$$\mathbf{x} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{bmatrix} [\mathbf{x}]_{\mathcal{B}} = -\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_4 = -1 + 2(2 + t) + 3(4 + 3t + 2t^2 + t^3)$$

It follows that $\mathbf{x} = 15 + 11t + 6t^2 + 3t^3$.

(5) 13. An eigenvector of the matrix $A = \begin{bmatrix} -1 & 2 & -3 \\ 3 & -2 & 1 \\ 0 & -2 & 0 \end{bmatrix}$ corresponding to $\lambda = 2$ is $\mathbf{v} = \begin{bmatrix} 105 \\ 63 \\ -63 \end{bmatrix}$. Compute $A\mathbf{v}$.

Solution: Since $A\mathbf{v} = \lambda\mathbf{v}$, $A\mathbf{v} = 2\mathbf{v} = \begin{bmatrix} 210 \\ 126 \\ -126 \end{bmatrix}$.

14. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$.

(3) (a) Find all of the eigenvalues of A.

Solution: The eigenvalues of A are the zeros of the characteristic polynomial for A.

$$\text{char } A = \lambda^2 - \text{tr } A\lambda + \det A = \lambda^2 - 8\lambda + ((2)(6) - (3)(-1)) = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5)$$

It follows that the eigenvalues of A are 3 and 5.

(4) (b) Find **an** eigenvector of A corresponding to **one** of the eigenvalues you found in (a).

Solution: For $\lambda = 3$, we want to find $\text{Nul}(A - 3I)$.

$$A - 3I = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

This leaves us with the relationship that $x_1 - 3x_2 = 0$ with x_2 being a free variable. Then $\mathbf{x} = x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so an eigenvector of A corresponding to $\lambda = 3$ is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(6) 15. Let $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}$. Given that an eigenvalue of A is $\lambda = 2$, find a basis for the eigenspace of A corresponding to $\lambda = 2$.

Solution: Let's find a basis for $\text{Nul}(A - 2I)$.

$$A - 2I = \begin{bmatrix} -1 & 1 & -2 \\ -1 & -3 & 2 \\ -1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & -4 & 4 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = -x_3$, $x_2 = x_3$, and $x_3 = x_3$, so a basis for the eigenspace corresponding to $\lambda = 2$ is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$