Visualizing Matrix Transformations

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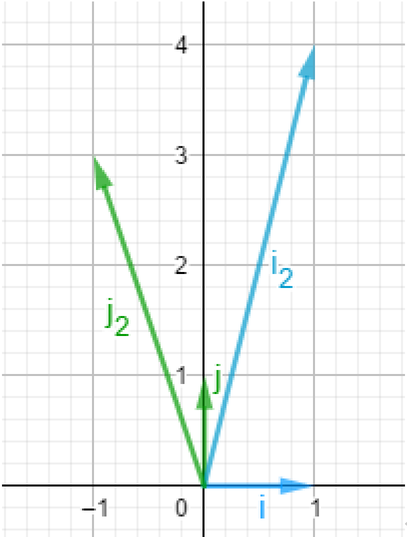
Who are we?

- Bre, applied mathematics major, transferring to PSU in the summer.
- Rocky, electrical engineering major, planning on transferring to PSU in the fall.

Our Angle of Attack

Recall that each column may be interpreted as the final resting point of the standard basis vectors.



Matrix Transformations in the Same Dimension

Reflection
Rotation
Shear
Scale

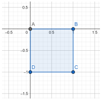


Figure: The unit square reflected about the x-axis.

General Standard Transformation Matrix

- $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is $A = \frac{1}{a^2+b^2}\begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}$
- $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is $A = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$

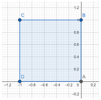


Figure: The unit square rotated through $\frac{\pi}{2}$.

General Standard Transformation Matrix

- $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is $\begin{bmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$
- $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is $A_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$

Shear

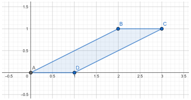


Figure: The unit square with a shear transformation applied.

General Standard Transformation Matrix

- $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$
- $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is $A = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

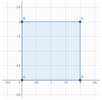


Figure: The unit square scaled by 2.

General Standard Transformation Matrix

- $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is $A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$
- $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

The Determinant

- For $T : \mathbb{R}^n \to \mathbb{R}^n, det(A) =$ factor of area/volume change.
- Since determinants can only be calculated for square matrices, $T : \mathbb{R}^m \to \mathbb{R}^n$ transformations are not applicable.

The Determinant in $\mathbb{R}^2$

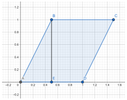


Figure: A parallelogram in $\mathbb{R}^2$.

- The base of the parallelogram is $\mathbf{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}$
- The height of the parallelogram is $\mathbf{h} = \begin{bmatrix} 0 \\ h \end{bmatrix}$

The Determinant in $\mathbb{R}^2$

A scale transformation is $A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$. It has $\det(A) = c_1 * c_2$

- $T(\mathbf{b}) = c_1 b$
- $T(\mathbf{h}) = c_2 h$

So, the transformed area is $T(A) = T(\mathbf{b}) * T(\mathbf{h}) = (c_1 c_2)bh = det(A)A$

The Determinant in $\mathbb{R}^3$

$V = bhw$. The transformation matrix will be $A = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$, with $\det(A) = c_1 c_2 c_3$.

- $\mathbf{b} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} \mapsto T(\mathbf{b}) = \begin{bmatrix} c_1 b \\ 0 \\ 0 \end{bmatrix}$
- $\mathbf{h} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix} \mapsto T(\mathbf{h}) = \begin{bmatrix} 0 \\ c_2 h \\ 0 \end{bmatrix}$
- $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} \mapsto T(\mathbf{w}) = \begin{bmatrix} 0 \\ 0 \\ c_3 w \end{bmatrix}$

So, the transformed volume is $T(V) = c_1 c_2 c_3 (bhw) = det(A)V$.

The Determinant

Example: $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ $det(A) = 4$

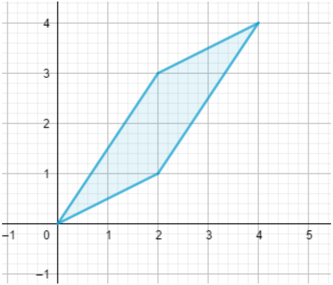


Figure: A parallelogram made of the standard basis vectors as vertices.

Isomorphic Planes

- To simply bring your basis vectors into R^m, pad your transformation matrix with zeros to give it the appropriate dimensions:
  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- In addition to the previously covered transformations, this opens up infinitely many new axes about which the image can be rotated.

Isomorphic Planes

- Rotation about the three main axes in R^3 is relatively simple. The standard rotational matrices for $\mathbb{R}^3 \to \mathbb{R}^3$ are used, with the third column removed.
- Notice the column corresponding to the axis of rotation remains unchanged.

$$T_x(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & -\sin\theta \\ 0 & \cos\theta \end{bmatrix}$$

$$T_y(\theta) = \begin{bmatrix} \cos\theta & 0 \\ 0 & 1 \\ -\sin\theta & 0 \end{bmatrix}$$

$$T_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \\ 0 & 0 \end{bmatrix}$$

Arbitrary Axis Rotation

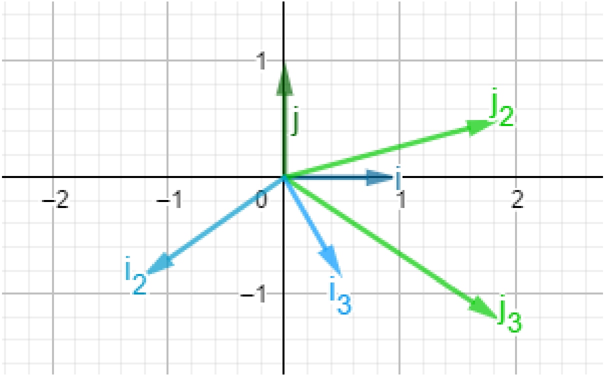
- Rotation around an arbitrary axis becomes trickier. Two pieces of information are needed:
  - Unit vector along your axis, $\mathbf{u} = (u_1, u_2, u_3)$
  - Angle of rotation, θ

$$\left[\begin{array}{ccc} \cos\theta + u_1^2(1-\cos\theta) & u_1u_2(1-\cos\theta) - u_3\sin\theta & u_1u_3(1-\cos\theta) + u_2\,\text{si} \\ u_2u_1(1-\cos\theta) + u_3\sin\theta & \cos\theta + u_2^2(1-\cos\theta) & u_2u_3(1-\cos\theta) - u_1\,\text{si} \\ u_3u_1(1-\cos\theta) - u_2\sin\theta & u_3u_2(1-\cos\theta) + u_1\sin\theta & \cos\theta + u_3^2(1-\cos\theta) \end{array}\right.$$

Combining Matrices

- Transformations may be stacked. A composite matrix may be found by multiplying $A \times B$.
- Order matters! This is consistent with the non-commutative property of matrix multiplication.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}$$



Recap

- 4 main types of transformations: rotation, reflection, scale, and shear.
- The determinant of a square standard transformation matrix tells you how much the area/volume of a shape will be stretched.
- You can use isomorphic planes to go between dimensions.
- A standard transformation matrix can contain more then one type of transformation.

Thank you for listening!

Any questions?