MTH 261 Final Exam Sample

Damien Adams

Sample Term

Name: ______________________________

For instructor use only
Score : /125

Some notes from Damien about the exam:

- You will have 2 hours 20 minutes to complete this exam. When you are finished, please turn it in to Damien.
- You may use a graphing calculator, but once you begin the exam, you *may not* use an internet sources or any other external source for assistance. This includes your notes, peers, textbook, tutors, other human beings, robots, or any other source I may have failed to mention.
- You may ask Damien for clarification on any problems, but you *may not* converse with anyone else regarding this exam.
- If you have any questions, ask Damien!
- The use of scratch paper is encouraged and should be included at the end of the exam. Please clearly label all work for each prompt.
- Please read all of the instructions carefully.
- Please show all of your work to each prompt. This is exceedingly important!
- Please use exact values unless otherwise stated.
- Please use vector notation any time you are writing a vector.
- Please show all row reduction steps.
- May the Force be with you!

Furthermore, please read all of the conditions below before beginning the exam. Indicate that you have read and understand these conditions by placing your initials on the line below.

- I have and will follow all of the guidelines stated above.
- I have not and will not cheat or violate any part of the Academic Integrity Policies.
- I will not provide or distribute any portion of this document to anyone other than the instructor.
- All work attached herein is my own authentic work.

Initials : ______________

(5) 1. Let $A = \begin{bmatrix} \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ A & K & Q & J \\ \flat & \sharp & \natural & ♩ \end{bmatrix}$. Suppose $\det A = 48$.

(a) Find $\det \begin{bmatrix} \flat & \sharp & \natural & ♩ \\ \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ A & K & Q & J \end{bmatrix}$.

(b) Find $\det \begin{bmatrix} -2\clubsuit & -2\diamondsuit & -2\heartsuit & -2\spadesuit \\ 0.1A & 0.1K & 0.1Q & 0.1J \\ \flat + 2A & \sharp + 2K & \natural + 2Q & ♩ + 2J \end{bmatrix}$.

(9) 2. Let $A \in M_{16\times 23}$ such that $\operatorname{rank} A = 15$. Find the following. *Show any arithmetic computations for partial credit consideration.*

(a) $\dim \operatorname{Row} A =$ _____

(b) $\dim \operatorname{Nul} A =$ _____

(c) $\dim \operatorname{LNul} A =$ _____

(9) 3. A vector space is a nonempty set V of vectors on which two operations are defined − addition (+) and scalar multiplication (×) − via ten axioms that must be true for all $\mathbf{u}, \mathbf{v} \in V$ and $c, d \in \mathbb{R}$. Which of the following are vector space **axioms**? (Circle all that apply)

(a) $1 \times \mathbf{v} = \mathbf{v}$

(b) $0 \times \mathbf{v} = \mathbf{0}$

(c) $c + \mathbf{v} \in V$

(d) $(cd) \times \mathbf{v} = c \times (d \times \mathbf{v})$

(e) $\mathbf{u} \times \mathbf{v} \in V$

(f) $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent

(g) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(h) $c \times (\mathbf{u} + \mathbf{v}) = (c \times \mathbf{u}) + (c \times \mathbf{v})$

(i) $(-1) \times \mathbf{u} = -\mathbf{u}$

(10) 4. Let $A \in M_{n\times n}$.

The Invertible Matrix Theorem is a massive theorem which states that a series of statements are equivalent. On the other hand, the theorem implies a Singular Matrix Theorem. Which of the following are equivalent to the statement A **is an invertible matrix.**? (Circle all that apply)

(a) $\operatorname{Nul} A$ consists of more than one vector

(b) $\operatorname{rank} A < n$

(c) The columns of A are linearly independent

(d) The equation $A\mathbf{x} = \mathbf{0}$ is inconsistent

(e) 0 is an eigenvalue of A

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(g) A is not square

(h) $\det A = 0$

(i) A^T is invertible

(j) A is row equivalent to I

(5) 5. Explain as specifically as possible what it means for a set $\mathcal{B}$ to be a **basis** for a vector space V.

6. Find the following determinants.

(3) (a) $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

(2) (b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 3 & 5 & 8 & 0 \\ 13 & 21 & 34 & 55 \end{vmatrix}$

(4) (c) $\begin{vmatrix} -1 & 2 & -3 \\ 3 & -2 & 1 \\ 0 & -2 & 0 \end{vmatrix}$

(10) 7. The set $GL_2(\mathbb{R})$ is the set of all invertible 2×2 matrices. Determine if $GL_2(\mathbb{R})$ is a subspace of $M_{2\times 2}$ or not. Show all work to support your conclusion.

8. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \in \mathbb{R}^3$, and let $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \end{bmatrix}$. Suppose RREF $A = \begin{bmatrix} 1 & 0 & 4 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(3) (a) Does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ span $\mathbb{R}^3$? Briefly justify your answer.

(3) (b) Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ linearly independent? If yes, briefly justify your answer. If no, **provide a linear dependence relation** among $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and $\mathbf{u}_4$.

(2) (c) Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ a basis for $\mathbb{R}^4$? Why or why not?

9. Let H be the subspace of $\mathbb{R}^3$ spanned by $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \right\}$.

(2) (a) Is S linearly independent or linearly dependent?

(3) (b) Find a basis for H consisting only of vectors in S.

(2) (c) Find $\dim H$.

(3) (d) Expand the basis you found in part (b) to a basis for $\mathbb{R}^3$.

10. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & -1 & 2 \\ 2 & 4 & 2 & 0 \end{bmatrix}$.

(5) (a) Find a basis for Col A.

(5) (b) Find a basis for Nul A.

(2) (c) rank $A =$ _____

(2) (d) nullity $A =$ _____

(2) (e) Col A is a subspace of $\mathbb{R}^m$ for $m =$ _____.

(2) (f) Nul A is a subspace of $\mathbb{R}^n$ for $n =$ _____.

11. Consider the polynomials $\mathbf{p}_1(t) = 1 + t + 2t^2$, $\mathbf{p}_2(t) = -t + 2t^2$, $\mathbf{p}_3(t) = 1 + 2t$, and $\mathbf{p}_4(t) = 1 + t + t^2$. Let $P = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$.

(3) (a) Does P span $\mathbb{P}_2$? Briefly justify your answer.

(3) (b) Is P linearly independent? If not, find a dependence relation for the polynomials.

(3) (c) If possible, find a basis for $\mathbb{P}_2$ consisting only of the vectors from $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$. If this is not possible, briefly state why.

(5) 12. Let $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = 2 + t$, $\mathbf{v}_3 = 3 + 2t + t^2$, and $v_4 = 4 + 3t + 2t^2 + t^3$. Then $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{P}_3$. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$, then find $\mathbf{x}$.

(5) 13. An eigenvector of the matrix $A = \begin{bmatrix} -1 & 2 & -3 \\ 3 & -2 & 1 \\ 0 & -2 & 0 \end{bmatrix}$ corresponding to $\lambda = 2$ is $\mathbf{v} = \begin{bmatrix} 105 \\ 63 \\ -63 \end{bmatrix}$. Compute $A\mathbf{v}$.

14. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$.

(3) (a) Find all of the eigenvalues of A.

(4) (b) Find **an** eigenvector of A corresponding to **one** of the eigenvalues you found in (a).

(6) 15. Let $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}$. Given that an eigenvalue of A is $\lambda = 2$, find a basis for the eigenspace of A corresponding to $\lambda = 2$.