10.5 Parametric Surfaces

10.5.1 Graphing a Parametric Surface

Previously, parametric equations were used to describe lines. Now, we will use them to describe surfaces.

Parametric equations of one parameter produce a one-dimensional graph (curve). Parametric equations of two parameters produce a two-dimensional graph (surface). Parametric equations of n parameters produce an n-dimensional graph.

Definition

The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v) \tag{1}$$

or

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \tag{2}$$

where $(u, v) \in D$, a region in the uv-plane. The graph, S, produced by either (1) or (2) is called a **parametric surface**. The equations in (1) are called **parametric equations** of S. The equation (2) is called a **vector equation** for S.
Note that the *interior* of the surface is not part of the graph.

Note: To plot a parametric surface in GeoGebra, use the command

```
S = Surface( x(u,v) , y(u,v) , z(u,v) , u , START , END , v , START , END )
```

where the first `START` and `END` represent the domain for u, and the second pair represents the domain of v.

Example 1. Let $\mathbf{r}(u, v) = \left\langle \ln u, \sqrt{2v}, \dfrac{2}{uv} \right\rangle$.

(a) Determine and sketch the domain of $\mathbf{r}$.

(b) Use GeoGebra to graph the surface described by $\mathbf{r}$.

10.5.2 Parametrizing a Surface

Example 2. Find a parametric representation for the part of the elliptic paraboloid $x + y^2 + 2z^2 = 4$ that lies above the plane $z = 0$.

Example 3. Let D be the cylinder $x^2 + y^2 = 4$ with $0 \leq z \leq 1$. Find a parametrization for D.

Example 4. Find a parametrization for the lower half of the sphere of radius 3 centered at the origin.

10.5.3 Tangent Planes to Parametric Surfaces

Suppose we want to linearize a surface S, where S is defined parametrically as

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

Let's say our point is found when $u = u_0$ and $v = v_0$. Then we have the point $P(x_0, y_0, z_0)$ where $x_0 = x(u_0, v_0)$, $y_0 = y(u_0, v_0)$, and $z_0 = z(u_0, v_0)$.

Notice

$$\mathbf{r}_u(u,v) = \left\langle \frac{\partial x}{\partial u}(u,v), \frac{\partial y}{\partial u}(u,v), \frac{\partial z}{\partial u}(u,v) \right\rangle \quad , \quad \mathbf{r}_v(u,v) = \left\langle \frac{\partial x}{\partial v}(u,v), \frac{\partial y}{\partial v}(u,v), \frac{\partial z}{\partial v}(u,v) \right\rangle$$

Then $\mathbf{r}_u(u,v)$ and $\mathbf{r}_v(u,v)$ are vectors on the tangent plane to S at the point determined by (u,v). Moreover, a normal vector to the tangent plane at P is

$$\mathbf{n}_{(u_0,v_0)} = \mathbf{r}_u(u_0,v_0) \times \mathbf{r}_v(u_0,v_0)$$

Example 5. Find the tangent plane to the parametric surface $\mathbf{r}(u,v) = \langle u^2, v^2, uv \rangle$ when $u = 1$ and $v = 1$.