SERIES SOLUTIONS TO DIFFERENTIAL EQUATIONS

MTH 253 LECTURE NOTES

Recall that we have learned how to solve separable equations, but this technique is quite limited to specific, "nice" differential equations where variables can be separated. More often than not, we will see differential equations that cannot be solved by those means. For example, $y'' - 2xy' + y = 0$ is not separable.

However, recall that functions can be represented as a power series, and we have learned how to differentiate power series. Perhaps we could find solutions to differential equations where $y = f(x)$ is represented as a power series.

Series Solutions to Differential Equations

The following is a problem-solving strategy for finding a power series solution to a differential equation.

1. Assume the differential equation has a solution of the following form.
$$y = \sum_{n=0}^{\infty} c_n x^n$$
2. Differentiate the power series term-by-term to get the following.
$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \quad = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$
3. Substitute the power series expressions into the differential equation.
4. Re-index sums as necessary to combine terms and simplify the expression.
5. Equate coefficients of like powers of x to determine values for the coefficients c_n in the power series.
6. Substitute the coefficients back into the power series and write the solution.

Example 1. Consider the differential equation $\frac{dy}{dx} - y = 0$.

a. Use power series to solve the differential equation. Does the solution look familiar?
b. Solve the differential equation using separation of variables to confirm.

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Example 2. Consider the differential equation $y'' + y = 0$.

a. Use power series to solve the differential equation. Do/does the power series look familiar? Use this to rewrite the solution with familiar functions.

b. Confirm that this is the solution to the differential equation by plugging the function, and its derivative(s), back into the differential equation.

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Note: This has been a brief overview of series solutions to differential equations. If you take a differential equations course, you will likely see a more rigorous treatment of this topic over the span of multiple sections.