FUNCTIONS & SURFACES

MTH 253 LECTURE NOTES

Definition

A **Function of Two Variables**, f, is a rule that assigns to each ordered pair (x, y) in a set D a unique number denoted $f(x, y)$. The set D is called the **Domain** of f, and its **Range** is the set of values that f attains: $\{f(x, y) \mid (x, y) \in D\}$. The variables x and y are called the **Independent Variables** of f.

Note: We will assume that the domain and range must contain only real numbers.

Note: Just like it is common to write $y = f(x)$, it is also common to write $z = f(x, y)$. In this case, the variable z is called the **Dependent Variable**. Notice D is a subset of $\mathbb{R}^2$, and the range is a subset of $\mathbb{R}$.

Definition

If f is a function of two variables with domain D, then the **Graph** of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$. The graph of a relationship of three variables is called a **Surface**.

Example 1. Consider the plane through the points $A(2, -1, 3)$, $B(1, -4, -2)$, and $C(0, 2, -1)$.

a. Find the linear equation of the plane.
b. Write a formula $z = f(x, y)$ for the plane.
c. Find $f(-1, 3)$.
d. Graph $z = f(x, y)$ in GeoGebra.
e. What is the domain of f?
f. What is the range of f?

Definition

A **Linear Function in Two Variables** is a function of the form $f(x,y) = ax+by+c$. The graph of a linear function in two variables is a plane.

Example 2. Let $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$. Verify your results in GeoGebra.

a. Find $f(4,11)$.

b. Find and sketch the domain of f.

Example 3. Let $f(x,y) = \sqrt{x} - \sqrt{y}$. Verify your results in GeoGebra.

a. Find $f(16,25)$.

b. Find and sketch the domain of f.

c. Find the range of f.

Exercise 1. Let $g(x,y) = x\ln(x^2 - y)$. Verify your results in GeoGebra.

a. Find $g(-1,2)$.

b. Find and sketch the domain of g.

c. Find the range of g.

Example 4. Draw a set of coordinate axes for $\mathbb{R}^3$. Sketch the graph of the following functions in $\mathbb{R}^3$ by hand on your axes. Verify your results in GeoGebra.

a. $f(x,y) = 4 - x^2$

b. $g(x,y) = \sin x$

Exploration: A linear equation is one of the form

$$ax + by + cz + d = 0.$$

The graph of such an equation is a plane. A quadratic equation is one of the form

$$ax^2 + bxy + cy^2 + dyz + ez^2 + fxz + gx + hy + iz + j = 0.$$

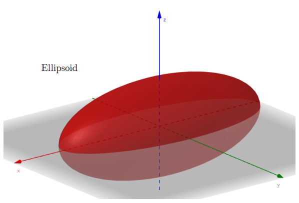
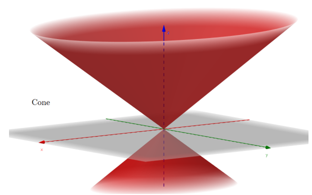
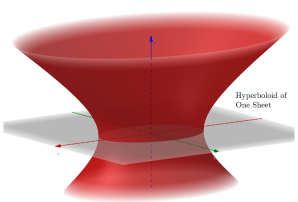
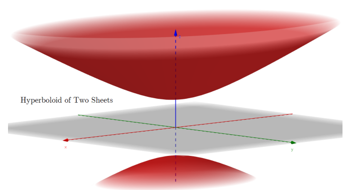
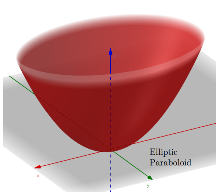
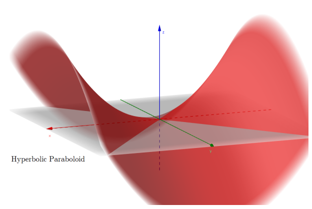
The graph of such an equation (if a, b, c, d, e, f are not all 0) is a **quadric surface**.

Definition

The graph of a quadratic equation in two variables is called a **Quadric Surface**.

Note: A quadric surface often does *not* correspond to an equation that represents a function.

Six basic quadric surfaces:

Surface	Equation	Surface	Equation
	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$		$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$
	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$		$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$		$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{z}{c}$

Exercise 2. Match each quadric surface with its appropriate equation.

a. $z = \dfrac{y^2}{4} - \dfrac{x^2}{3}$

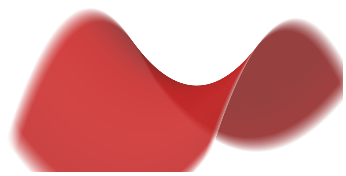
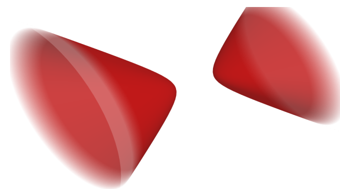
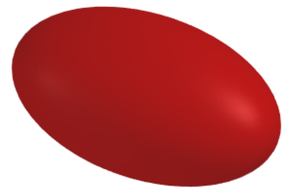
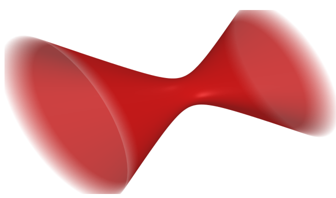
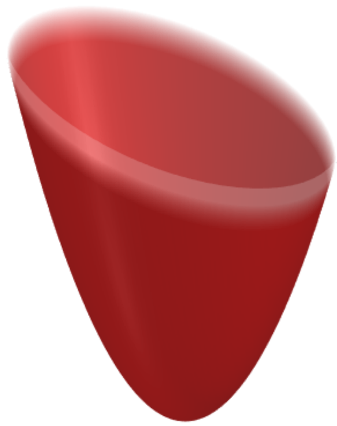
b. $\dfrac{x^2}{4} + z^2 - \dfrac{y^2}{4} = 1$

c. $\dfrac{z}{4} = \dfrac{x^2}{8} + \dfrac{y^2}{2}$

d. $-x^2 + y^2 - 4z^2 = 1$

e. $\dfrac{z^2}{4} = x^2 + \dfrac{y^2}{4}$

f. $\dfrac{x^2}{4} + y^2 + z^2 = 1$

—		—		—	
—		—		—