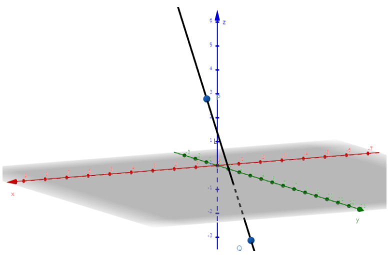
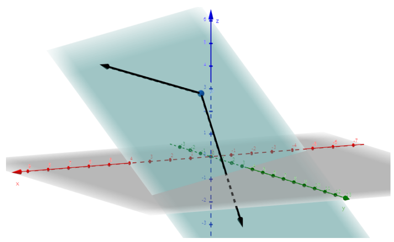
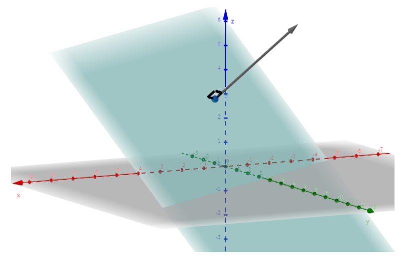
PLANES

MTH 253 LECTURE NOTES

Exploration: We will explore three different ways to describe a plane in 3-space.

A line is determined either by two points or a point and a direction. A plane is determined by either three points or a point and two directions. However, instead of two separated directions, we can instead substitute those two directions with a single *orthogonal* direction.

Definition

A vector $\mathbf{n}$ orthogonal to a plane P is called a **Normal Vector** to P.

Definition

Let $Q(x_0, y_0, z_0)$ be a point on a plane in $\mathbb{R}^3$. If $P(x, y, z)$ is an arbitrary point on the plane, $\mathbf{r_0} = \langle x_0, y_0, z_0\rangle$ is the position vector for Q, and $\mathbf{r} = \langle x, y, z\rangle$ is the position vector for P, then a vector in the plane $\overrightarrow{QP}$ is given by

$$\mathbf{r} - \mathbf{r_0} = \langle x - x_0, y - y_0, z - z_0\rangle.$$

Moreover, if $\mathbf{n} = \langle a, b, c\rangle$ is a normal vector to the plane, then

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0 \qquad \iff \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$$

are each called a **Vector Equation of the Plane**.

Exploration: Since

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) &= 0 \\ \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \end{aligned}$$

Definition

The equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

is called the **Scalar Equation of the Plane Through** $Q(x_0, y_0, z_0)$ **with Normal Vector** $\langle a, b, c\rangle$.

Example 1. Find both the vector equation and the scalar equation of the plane through $P(1, 2, 3)$ with normal vector $\mathbf{n} = \langle 2, -1, 5 \rangle$. Find the intercepts of the plane, then sketch the plane.

Exploration: If we expand the scalar equation, we obtain a new form of an equation for the plane.

Definition

An equation of the form

$$ax + by + cz + d = 0$$

is called a **Linear Equation** in x, y, z. If a, b, c are not all 0, then the linear equation represents a plane with normal vector $\langle a, b, c \rangle$.

Example 2. Find a linear equation for the plane through $P(1, 2, 3)$ with normal vector $\mathbf{n} = \langle 2, -1, 5 \rangle$. Then graph the plane in GeoGebra.

Example 3. Find the point at which the line with parametric equations

$$x = 1 + 2t \quad , \quad y = 3 - t \quad , \quad z = -4 + 5t$$

intersects the plane $17x + 11y + 8z - 63 = 0$.

Exercise 1. Let Π be the plane determined by the points $P(1,2,3)$, $Q(2,-1,5)$, and $R(4,1,-2)$.

a. Find a vector equation for Π.
b. Find a scalar equation for Π.
c. Find a linear equation for Π.
d. Find the intercepts of Π and sketch the plane by hand.
e. Find the point at which the line whose vector equation is

$$\langle x,y,z\rangle = \langle 1,3,-4\rangle + t\langle 2,-1,5\rangle$$

intersects Π.
f. Graph Π in GeoGebra.

Definition

Two distinct planes in $\mathbb{R}^3$ are **Parallel** if the planes do not intersect. This occurs precisely when their normal vectors are parallel.

Theorem

Two planes in $\mathbb{R}^3$ that intersect are either coincident or intersect in a line. The angle between the planes is the same as the angle between their normal vectors.

Example 4. The two planes $3x-14y-19z+14=0$ and $ax+by+cz-4=0$ are parallel. Find a,b,c.