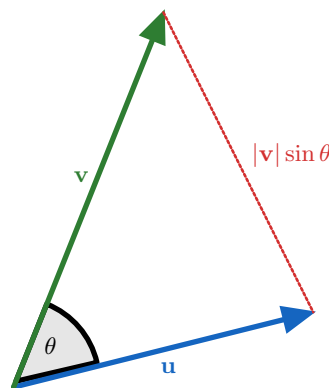
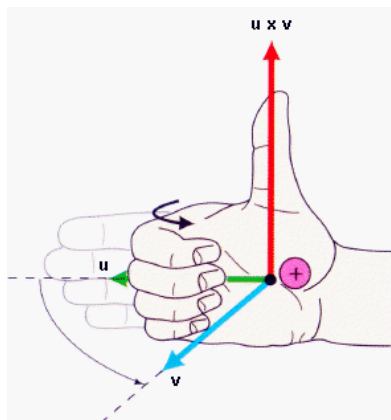


CROSS PRODUCT

MTH 253 LECTURE NOTES

Exploration: The **cross product** of two vectors \mathbf{u} and \mathbf{v} in V_3 is another vector in V_3 , called $\mathbf{u} \times \mathbf{v}$, that is orthogonal to both \mathbf{u} and \mathbf{v} . The length of this vector depends entirely upon both the lengths of \mathbf{u} and \mathbf{v} as well as the angle between them, θ .

Given two vectors \mathbf{u} and \mathbf{v} in V_3 with the same initial point, you may notice that there are two different directions that $\mathbf{u} \times \mathbf{v}$ could point. The appropriate direction is determined by the right-hand rule.



Definition

Let \mathbf{u} and \mathbf{v} be nonzero vectors in V_3 . Then the **Cross Product** of \mathbf{u} and \mathbf{v} is the vector $\mathbf{u} \times \mathbf{v}$ in V_3 such that

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin \theta \mathbf{n}$$

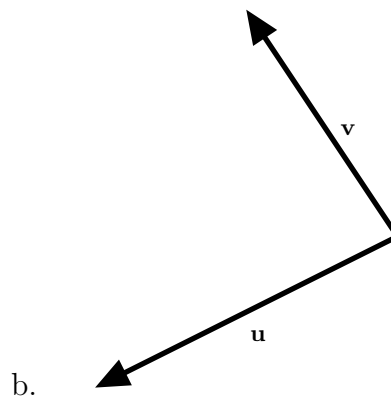
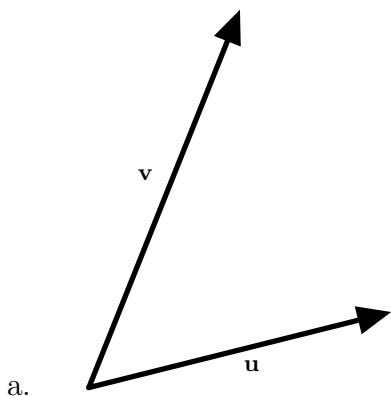
where \mathbf{n} is the unit vector orthogonal to \mathbf{u} and \mathbf{v} whose direction is determined by the right-hand rule. The vector \mathbf{n} is called the **Normal Vector**.

Note: The cross product is defined *only* for vectors in V_3 and produces another vector in V_3 .

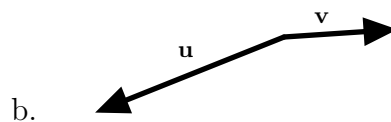
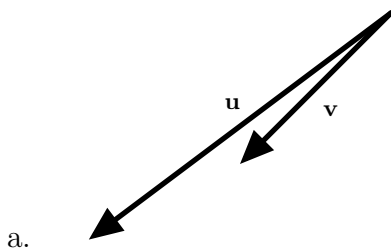
Theorem

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

Example 1. Given the vectors \mathbf{u} and \mathbf{v} , determine if $\mathbf{u} \times \mathbf{v}$ goes “out of the page” or “into the page”.



Exercise 1. Given the vectors \mathbf{u} and \mathbf{v} , determine if $\mathbf{u} \times \mathbf{v}$ goes “out of the page” or “into the page”.



Exploration: What are some observations about the cross product?

Example 2. Let $\mathbf{u} = \langle 2, 4, 0 \rangle$ and $\mathbf{v} = \langle 1, 7, 0 \rangle$.

a. Find $\mathbf{u} \times \mathbf{v}$.

b. Use GeoGebra to graph \mathbf{u} and \mathbf{v} .

c. Use GeoGebra to graph $\mathbf{u} \times \mathbf{v}$ using `Cross((2,4,0), (1,7,0))`. Visualize the vectors, the plane formed by the vectors, and the resulting cross product.

Exercise 2. Let $\mathbf{u} = \langle 2, 4, 0 \rangle$ and $\mathbf{v} = \langle 1, 7, 0 \rangle$. Find $\mathbf{v} \times \mathbf{u}$, then confirm your result using GeoGebra.

Example 3. Compute the following cross products, assuming they are V_3 vectors.

a. $\mathbf{i} \times \mathbf{j}$

b. $\mathbf{j} \times \mathbf{i}$

c. $\mathbf{i} \times \mathbf{k}$

Exercise 3. Compute the following cross products, assuming they are V_3 vectors.

a. $\mathbf{k} \times \mathbf{i}$

b. $\mathbf{j} \times \mathbf{k}$

c. $\mathbf{k} \times \mathbf{j}$

Cross Product Properties

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V_3$ are nonzero and $c \in \mathbb{R}$, then

i. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

iv. $\mathbf{v} \times \mathbf{0} = \mathbf{0} \times \mathbf{v} = \mathbf{0}$

ii. $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v})$

v. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

iii. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

vi. $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

Definition

Component Definition of the Cross Product:

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be two nonzero vectors in V_3 . The **Cross Product** (or **Vector Product**) of two vectors \mathbf{u} and \mathbf{v} , written $\mathbf{u} \times \mathbf{v}$, is the V_3 vector

$$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1 \rangle$$

Proof: We claim that the component definition of the cross product gives a vector orthogonal to

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors in V_3 . Write $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$.

If \mathbf{u} and \mathbf{c} are orthogonal, then $\mathbf{u} \cdot \mathbf{c} = 0$. If \mathbf{v} and \mathbf{c} are orthogonal, then $\mathbf{v} \cdot \mathbf{c} = 0$. Then

$$u_1c_1 + u_2c_2 + u_3c_3 = 0 \tag{1}$$

$$v_1c_1 + v_2c_2 + v_3c_3 = 0 \tag{2}$$

If we multiply (1) by v_3 and (2) by $-u_3$, then we get

$$u_1v_3c_1 + u_2v_3c_2 + u_3v_3c_3 = 0 \tag{3}$$

$$-u_3v_1c_1 - u_3v_2c_2 - u_3v_3c_3 = 0 \tag{4}$$

Adding (3) and (4), we get

$$(u_1v_3 - u_3v_1)c_1 + (u_2v_3 - u_3v_2)c_2 = 0 \quad (5)$$

If

$$c_1 = u_2v_3 - u_3v_2 \quad \text{and} \quad c_2 = -(u_1v_3 - u_3v_1)$$

then we have a legitimate solution. And if we substitute these into (1), we get

$$c_3 = u_1v_2 - u_2v_1$$

It follows that the resulting vector

$$\mathbf{c} = \langle u_2v_3 - u_3v_2, -u_1v_3 + u_3v_1, u_1v_2 - u_2v_1 \rangle$$

is orthogonal to both \mathbf{u} and \mathbf{v} . □

Example 4. Use the formula above to confirm $\mathbf{u} \times \mathbf{v}$ for the vectors $\mathbf{u} = \langle 2, 4, 0 \rangle$ and $\mathbf{v} = \langle 1, 7, 0 \rangle$.

Exercise 4. Use the formula above to confirm $\mathbf{v} \times \mathbf{u}$ for the vectors $\mathbf{u} = \langle 2, 4, 0 \rangle$ and $\mathbf{v} = \langle 1, 7, 0 \rangle$.

Note: Though this formula is true, this is not an effective way to compute $\mathbf{u} \times \mathbf{v}$. Moreover, the original definition of the cross product requires us to know θ , and we may not have that information. The following information about matrices and determinants will cover topics from Linear Algebra very briefly and without justification. For more clarity, it is suggested to take Linear Algebra.

Definition

A **Matrix** is a rectangular array of numbers. For example, a 2×2 matrix looks like

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, \text{ and a } 3 \times 3 \text{ matrix looks like } \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Definition

The **Determinant** of a 2×2 matrix $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is given by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Example 5. Calculate the determinant $\begin{vmatrix} -6 & -4 \\ 1 & -3 \end{vmatrix}$. Confirm your result in GeoGebra using `Determinant({{-6,-4},{1,-3}})`.

Exercise 5. Calculate the determinant $\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$.

Definition

The **Determinant** of a 3×3 matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Note: Here is a neat trick for remembering this.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 6. Calculate the determinant $\begin{vmatrix} 2 & 3 & 1 \\ -3 & 6 & 4 \\ -5 & -1 & 3 \end{vmatrix}$. Confirm your result in GeoGebra.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are nonzero vectors in V_3 , then

$$\mathbf{u} \times \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Note: This is a *complete* abuse of notation, just like thinking of $\frac{dy}{dx}$ as a fraction. However, this methodology will suffice for the purposes of computation, and it is common to do so.

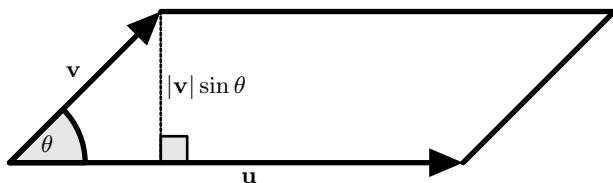
Example 7. Let $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 2, -1, 5 \rangle$. Find $\mathbf{u} \times \mathbf{v}$ by hand. Confirm your conclusion in GeoGebra.

Theorem

Two nonzero vectors \mathbf{u} and \mathbf{v} are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Proof:

Exploration: The area of a parallelogram is given by base times height. So for the area of the parallelogram on two vectors \mathbf{u} and \mathbf{v} with the same initial point with θ being the smallest angle between them, we get $A = |\mathbf{u}||\mathbf{v}|\sin\theta$.



On the other hand, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin\theta||\mathbf{n}|$. Since $|\mathbf{n}| = 1$, and because $\sin\theta \geq 0$, we get

Theorem

The area of the parallelogram determined by the V_3 vectors \mathbf{u} and \mathbf{v} is $|\mathbf{u} \times \mathbf{v}|$.

Note: The area of the *triangle* determined by \mathbf{u} and \mathbf{v} would be $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$.

Example 8. Find the area of the parallelogram formed by $\langle 1, -2, 4 \rangle$ and $\langle -3, 6, 7 \rangle$.

Exercise 6. Find the area of the parallelogram formed by $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 2, -1, 5 \rangle$.

Example 9. Find a unit vector perpendicular to the plane that passes through the points $A(2, -5, -6)$, $B(2, -4, -4)$, and $C(-1, -7, -5)$, then find the area of the triangle formed by points A, B, C .

Exercise 7. Find a unit vector perpendicular to the plane that passes through the points $U(4, 1, -2)$, $V(-1, 2, 4)$, and $W(2, -3, 3)$.

Note: Cross products show up in physics, notably in rotational mechanics where the torque is the cross product of the radial vector with the force vector: $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. We will leave this application to the physicists.