DOT PRODUCT

MTH 253 LECTURE NOTES

Definition

The **Dot Product** (or **Inner Product** or **Scalar Product**) of two nonzero vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is the number

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example 1. Let $\mathbf{u} = \langle -3, 4, 2 \rangle$ and $\mathbf{v} = \langle 2, -1, 5 \rangle$. Find $\mathbf{u} \cdot \mathbf{v}$.

Exercise 1. Let $\mathbf{v} = \langle 2, -1, 5 \rangle$ and $\mathbf{w} = \langle -4, 2, -10 \rangle$. Find $\mathbf{v} \cdot \mathbf{w}$.

Properties of the Dot Product

If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V_3$ and $c \in \mathbb{R}$, then

i. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \geq 0$

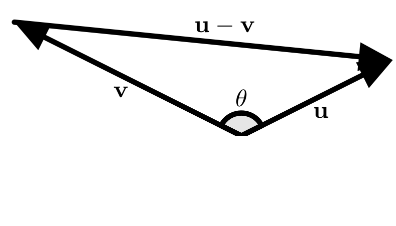
ii. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

iii. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

iv. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

v. $\mathbf{0} \cdot \mathbf{u} = 0$

Exploration: Consider the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Use the Law of Cosines to find a relationship between $\mathbf{u}$, $\mathbf{v}$, and θ.



Theorem

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, where $\theta \in [0, \pi]$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Corollary

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$.

Example 2. Let $\mathbf{u} = \langle -3, 4, 2\rangle$ and $\mathbf{v} = \langle 2, -1, 5\rangle$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

Example 3. Let $\mathbf{u} = \langle -3, 4, 2\rangle$ and $\mathbf{w} = \langle -4, 2, -10\rangle$. Find the angle between $\mathbf{u}$ and $\mathbf{w}$.

Exercise 2. Let $\mathbf{v} = \langle 2, -1, 5\rangle$ and $\mathbf{w} = \langle -4, 2, -10\rangle$. Find the angle between $\mathbf{v}$ and $\mathbf{w}$.

Definition

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are called **Orthogonal** (or **Perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$.
Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are called **Parallel** if $\mathbf{u} = c\mathbf{v}$ where $c \in \mathbb{R}$.

Exercise 3. Let $\mathbf{u} = \langle -3, 4, 2\rangle$, $\mathbf{v} = \langle 2, -1, 5\rangle$, and $\mathbf{w} = \langle -4, 2, -10\rangle$. Determine which vectors are parallel or orthogonal.

Exploration: Consider two vectors $\mathbf{u}$ and $\mathbf{v}$ with the same initial point P. Given these two vectors, we may want to find the projection of one onto the other.

Definition

Let $\mathbf{u}, \mathbf{v} \in V_n$. If we drop a perpendicular from $\mathbf{u}$ onto the line on $\mathbf{v}$ and call the point of intersection Q, then we can define two projections.

- The **Vector Projection of v onto u**, $\text{proj}_{\mathbf{u}}\, \mathbf{v}$, is the vector $\overrightarrow{PQ}$.
- The **Scalar Projection of v onto u**, $\text{comp}_{\mathbf{u}}\, \mathbf{v}$, is the signed magnitude of $\text{proj}_{\mathbf{u}}\, \mathbf{v}$, where the quantity is positive if $\text{proj}_{\mathbf{u}}\, \mathbf{v}$ is in the same direction as $\mathbf{u}$ and negative if in the opposite direction.

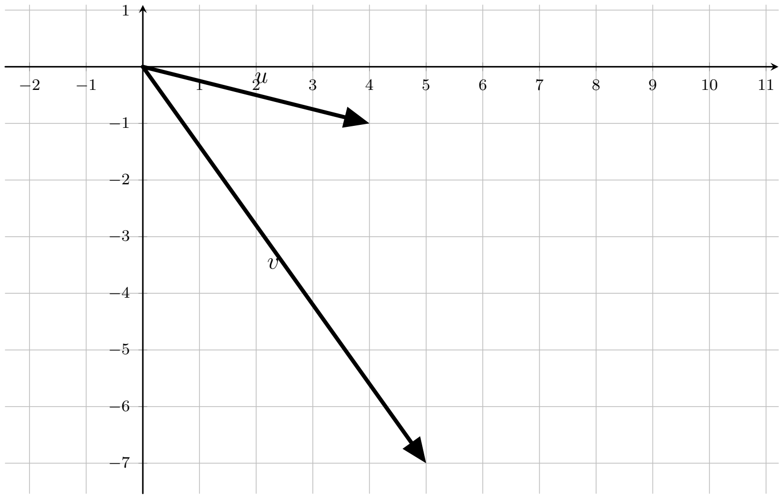
Exploration: Let's find nice formulas for $\text{comp}_{\mathbf{u}}\, \mathbf{v}$ and $\text{proj}_{\mathbf{u}}\, \mathbf{v}$ that involve $\mathbf{u}$ and $\mathbf{v}$.

Theorem

If $\mathbf{u}, \mathbf{v} \in V_n$, then

$$\begin{aligned} \text{comp}_{\mathbf{u}}\, \mathbf{v} &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \\ \text{proj}_{\mathbf{u}}\, \mathbf{v} &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2}\mathbf{u} \\ &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \end{aligned}$$

Example 4. Let $\mathbf{u} = \langle 4, -1 \rangle$ and $\mathbf{v} = \langle 5, -7 \rangle$.



a. Find $\text{comp}_{\mathbf{u}}\, \mathbf{v}$ and $\text{proj}_{\mathbf{u}}\, \mathbf{v}$.

b. Find $\text{comp}_{\mathbf{v}}\, \mathbf{u}$ and $\text{proj}_{\mathbf{v}}\, \mathbf{u}$.

Exercise 4. Let $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 2, -1, 5 \rangle$. Find $\text{comp}_{\mathbf{u}}\, \mathbf{v}$ and $\text{proj}_{\mathbf{u}}\, \mathbf{v}$.