VECTORS

MTH 253 LECTURE NOTES

Definition

A **Vector** is a quantity that has both size (**Magnitude**) and direction. We often represent a vector as a directed line segment – an arrow with an **Initial Point** (or **Tail**) and a **Terminal Point** (or **Tip**). We write vectors with either bold font (whenever it is typed) or with an arrowhead (whenever it is handwritten).

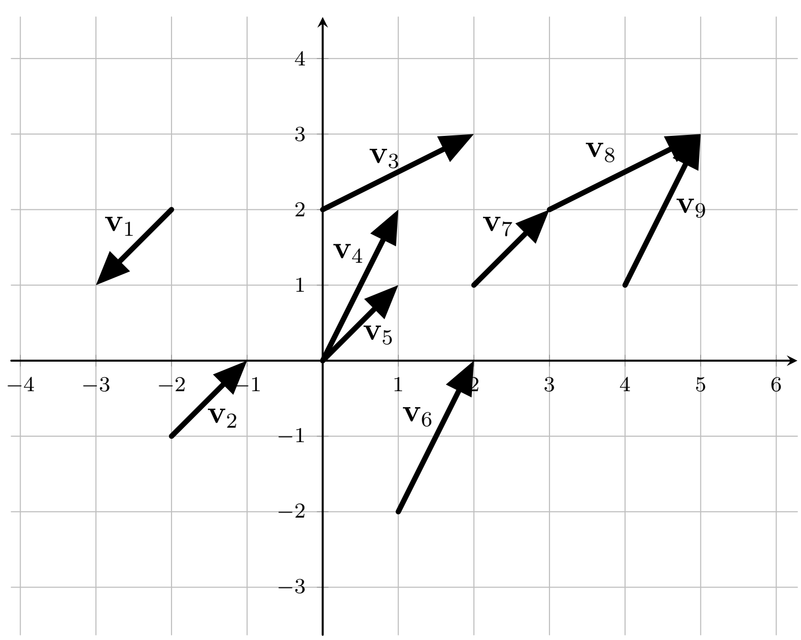
Exploration: Let's draw vectors $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{CD}$ for some points A, B, C, D.

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **Equivalent Vectors** if they have the same size and direction.

Note: Equivalent vectors do not need to have the same initial and terminal points.

Example 1. Below are nine vectors, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9$. Determine which vectors are equivalent.



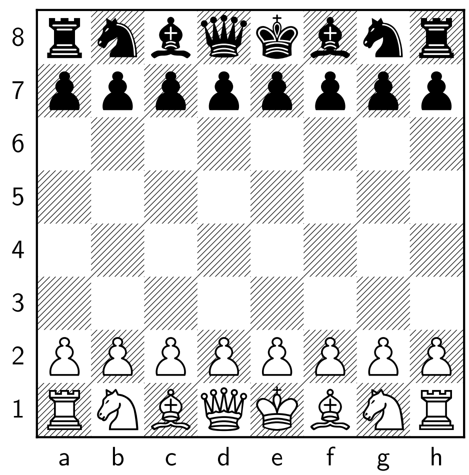
Definition

The vector whose length is zero and has no direction is called the **Zero Vector**, denoted $\mathbf{0}$.

Definition

A vector whose initial point and terminal point represent linear motion is called a **Displacement Vector**.

Exploration: For example, consider the chessboard below. Let's draw displacement vectors for the movement of several pieces.

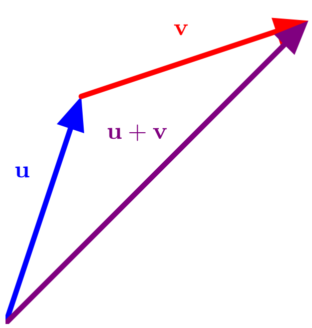


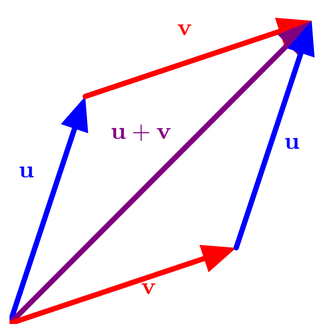
Definition

If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so that the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the **Sum of u and v** is the vector $\mathbf{u}+\mathbf{v}$ whose initial point is the initial point of $\mathbf{u}$ and whose terminal point is the terminal point of $\mathbf{v}$.

Note: We think of vector addition as the combination of displacements. More broadly, we have two addition laws called the Triangle Law of Addition and the Parallelogram Law of Addition that show graphical vector addition.

Triangle and Parallelogram Laws of Addition: Tail-to-Tip addition.

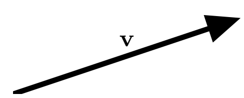




Definition

If c is a **Scalar** (i.e. a constant) and $\mathbf{v}$ is a vector, then the **Scalar Multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same if $c > 0$ and opposite if $c < 0$. If $c = 0$, then $c\mathbf{v} = \mathbf{0}$.
We call $-\mathbf{v} = (-1)\mathbf{v}$ the **Negative of v**.

Example 2. Given the vector $\mathbf{v}$ below, draw and label $2\mathbf{v}$, $3\mathbf{v}$, $\frac{1}{2}\mathbf{v}$, $-\mathbf{v}$, and $-3\mathbf{v}$.



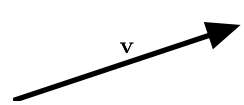
Definition

Two vectors are **Parallel** if they are scalar multiples.

Note: Parallel vectors have the same direction; parallel vectors may differ in size.

Example 3. Given the vector $\mathbf{v}$ below, graph three vectors that are parallel to $\mathbf{v}$ with the following conditions.

i. One is in the same direction as $\mathbf{v}$.
ii. One is in the opposite direction of $\mathbf{v}$.
iii. One has its initial point at the terminal point of $\mathbf{v}$.



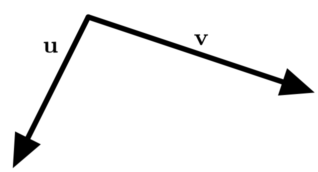
Definition

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, the **Difference of u and v** is the vector $\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v})$.

Note: In order to draw $\mathbf{u} - \mathbf{v}$, we first draw $-\mathbf{v}$, then we use the Triangle of Parallelogram Law of Addition.

Exercise 1. Below are two vectors $\mathbf{u}$ and $\mathbf{v}$. Draw and label the following vectors. You may wish to redraw $\mathbf{u}$ and $\mathbf{v}$ several times in order to draw these vectors.

a. $2\mathbf{u}$ b. $-3\mathbf{v}$ c. $\mathbf{u} + \mathbf{v}$ d. $\mathbf{v} - \mathbf{u}$ e. $\mathbf{u} - 2\mathbf{v}$



Note: Everything that we have done so far has been with graphical definitions, but it has not involved algebra. If we happen to know locations and had some numerical values to represent the magnitude and direction of the vector, then we will see that it mostly works intuitively.

Definition

If $\mathbf{u}$ has its tail at the origin and its tip at the point (u_1, u_2) or (u_1, u_2, u_3), then the vector $\mathbf{u}$ is said to have **Coordinates** (u_1, u_2) or (u_1, u_2, u_3). We express the vector in **Component Form** as

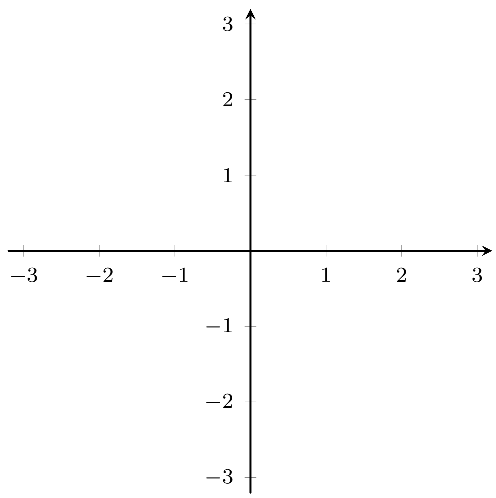
$$\mathbf{u} = \langle u_1, u_2 \rangle \qquad \text{or} \qquad \mathbf{u} = \langle u_1, u_2, u_3 \rangle$$

Definition

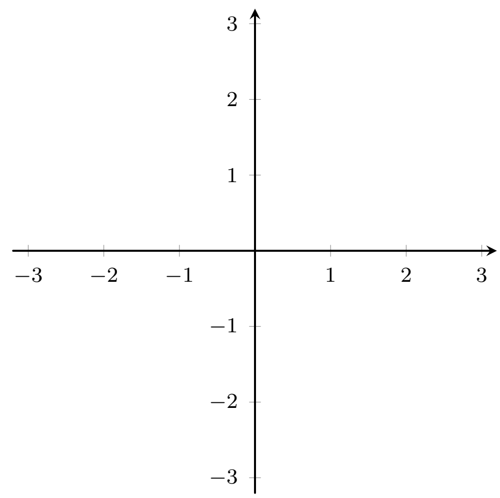
The graph of a vector of a fixed size and direction which has any particular initial point is called a **Representation** of a vector. The representation of a vector with initial point at the origin is called a **Position Vector**.
Given two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{u}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{u} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 4. Graph the vector $\mathbf{u} = \langle 2, -1 \rangle$ with four different representations, one of which is as a position vector. Then use Geogebra to graph two of these representations.



Example 5. Graph the points $A(3, 2)$ and $B(-2, -1)$, then graph the vector $\mathbf{u}$ whose representation is $\overrightarrow{AB}$. Write the component form of $\mathbf{u}$. Then use Geogebra to graph $\mathbf{u}$.



Exercise 2. Let $A(1, 2, 3)$ and $B(2, -1, 5)$ be points in $\mathbb{R}^3$. What is the vector whose representation is $\overrightarrow{AB}$? Graph this vector in Geogebra.

Definition

Given a vector $\mathbf{u}$, the **Magnitude** (or **Length** or **Norm**) of $\mathbf{u}$ is the length of any representation of $\mathbf{u}$, denoted with either $|\mathbf{u}|$ or $\|\mathbf{u}\|$.

Example 6. Find the length of $\mathbf{w} = \langle 1, 2 \rangle$.

Length of a Vector

If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2} \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Example 7. Find the length of $\mathbf{u} = \langle 1, 2, 3 \rangle$.

Definition

An n-dimensional vector is an ordered n-tuple $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$. Addition and scalar multiplication of n-dimensional vectors are defined just as with 2- and 3-dimensional vectors.

Component Vector Algebra

To add, subtract, and scale vectors given in component form, we add, subtract, and scale components, respectively. That is, if $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$, $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$, and $c \in \mathbb{R}$, then

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle \\ \mathbf{u} - \mathbf{v} &= \langle u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n \rangle \\ c\mathbf{u} &= \langle cu_1, cu_2, \ldots, cu_n \rangle \end{aligned}$$

Exercise 3. Let $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 2, -1, 5 \rangle$. Find the following

a. $\mathbf{u} + \mathbf{v}$

b. $\mathbf{v} - \mathbf{u}$

c. $2\mathbf{u} + 3\mathbf{v}$

d. $|2\mathbf{u} + 3\mathbf{v}|$

Definition

We denote V_n to be the set of all n-dimensional vectors. That is, V_2 is the set of all 2-dimensional vectors, and V_3 is the set of all 3-dimensional vectors.

Theorem

Properties of Vectors:
If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V_n$ and $c, d \in \mathbb{R}$, then

i. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

ii. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

iii. $\mathbf{u} + \mathbf{0} = \mathbf{u}$

iv. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

v. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

vi. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

vii. $(cd)\mathbf{u} = c(d\mathbf{u})$

viii. $1\mathbf{u} = \mathbf{u}$

Definition

A **Unit Vector** is a vector whose length is 1. If $\mathbf{u} \neq \mathbf{0}$, then the unit vector in the same direction of $\mathbf{u}$ is $\dfrac{1}{|\mathbf{u}|}\mathbf{u} = \dfrac{\mathbf{u}}{|\mathbf{u}|}$. The process of constructing the unit vector in the same direction of $\mathbf{u}$ is called **Normalizing u**.

Definition

The unit vectors $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$ in V_2 are called the **Standard Basis Vectors for** V_2. The unit vectors $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$ in V_3 are called the **Standard Basis Vectors for** V_3.

Example 8. Decompose $\mathbf{u} = \langle 1, 2, 3\rangle$ into terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

Exercise 4. Decompose $\mathbf{v} = \langle 2, -1, 5\rangle$ into terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

Note: We can now generalize and conclude that

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Example 9. Find a unit vector in the opposite direction as $\mathbf{v} = \langle 2, -1, 5\rangle$. Express this vector in terms of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

Exercise 5. Normalize $\mathbf{w} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Express this vector in component form.