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## LESSON 16

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# Double Integrals over Polar Regions

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## 16.1 Introduction

Some regions are most easily described using polar coordinates. For example, a disk with radius 2 in rectangular coordinates has the equation  $x^2 + y^2 \leq 4$  and we cannot solve for  $x$  or  $y$  as explicit functions of the other variable (we can write  $y = \pm\sqrt{4 - x^2}$  but this is not a function). On the other hand, in polar coordinates we can describe the disk of radius 2 simply as  $r \leq 2$ .

To further highlight this, the region in the first quadrant and inside the spiral  $r \leq 3 - \theta$  would need to be described as  $\sqrt{x^2 + y^2} \leq 3 - \tan^{-1}(y/x)$  if we wanted to use rectangular coordinates. This region can be seen in figure 16.1.1 to the right.

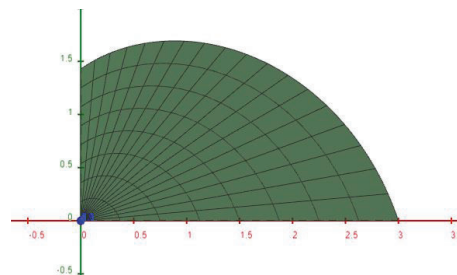


Figure 16.1.1: Region inside QI and the Spiral  $r \leq 3 - \theta$

View in Geogebra:

<https://www.geogebra.org/3d/ppgdnr7j>

So, if we wanted to find the volume under the surface  $f(x, y) = x + y + 1$  over the region in the first quadrant and inside the spiral  $r \leq 3 - \theta$ , it is easiest to do so in polar coordinates.

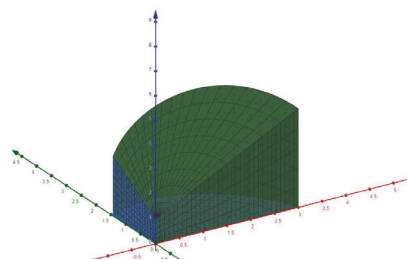


Figure 16.1.2: Volume inside QI, the Spiral  $r \leq 3 - \theta$ , and under  $z = x + y + 1$

View in Geogebra:

<https://www.geogebra.org/3d/ppgdnr7j>

## 16.2 Setting up a Double Integral Using Polar Coordinates

To find volumes using polar coordinates we will use the double integral

$$\int_{\alpha}^{\beta} \int_{r_0}^{r_f} f(r \cos(\theta), r \sin(\theta)) \cdot r \, dr \, d\theta$$

where  $r_0$  and  $r_f$  are bounds on the radius of the region we are integrating over and  $\alpha$  and  $\beta$  are bounds on the angle,  $\theta$  over which we are integrating. **Note the extra  $r$  in the formula before the  $dr \, d\theta$ !** Let's look at an example of using this before proving that this will give us the volume that we are looking for.

**Example 16.2.1** Evaluate  $\iint_D xy^2 \, dA$  where  $D$  is the region enclosed in the unit circle and to the right of the  $y$ -axis, using polar coordinates.

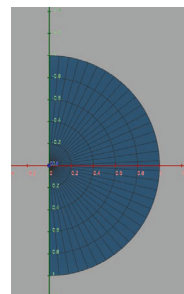


Figure 16.2.1: Region inside the Unit Circle with  $y \geq 0$

View in Geogebra:

<https://www.geogebra.org/3d/qqj3agwz>

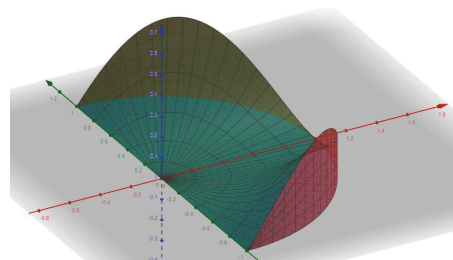


Figure 16.2.2: Volume inside the Unit Circle with  $y \geq 0$  and  $0 \leq z \leq xy^2$

View in Geogebra:

<https://www.geogebra.org/3d/qqj3agwz>

**Exercise 16.2.1** Evaluate  $\iint_D 1 - x^2 - y^2 \, dA$  where  $D$  is the region inside the unit circle.

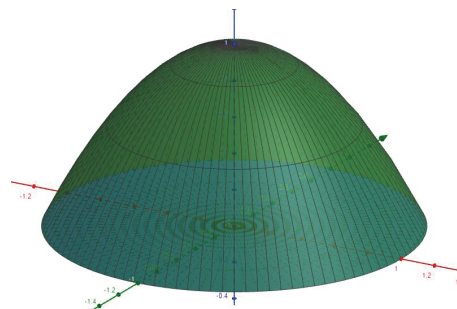


Figure 16.2.3: Volume inside

$$0 \leq z \leq 1 - x^2 - y^2$$

View in Geogebra:

<https://www.geogebra.org/3d/nuvn35js>

## 16.3 Formulating the Polar Double Integral

Suppose we are looking for the volume between the  $xy$ -plane and a surface,  $z = f(x, y)$ , inside some region,  $D$ , described in polar coordinates with  $\alpha \leq \theta \leq \beta$  and  $r_0 \leq r \leq r_f$ . Let's build up the double integral in polar coordinates by slicing the region we are finding the volume over every  $\Delta\theta$  and every  $\Delta r$ , finding the area,  $A_{i,j}$  of these slices, multiplying each area by a  $z_{i,j} = f(x_i, y_j)$ -value where  $(x_i, y_j)$  is in the slice whose volume we are trying to find so that  $V_{i,j} = z_{i,j} \cdot A_{i,j}$ . We see this in figure 16.3.1 below where the total volume of the solid can be approximated by adding up the volumes of each of the prisms acquired after slicing the region we are finding the volume over.

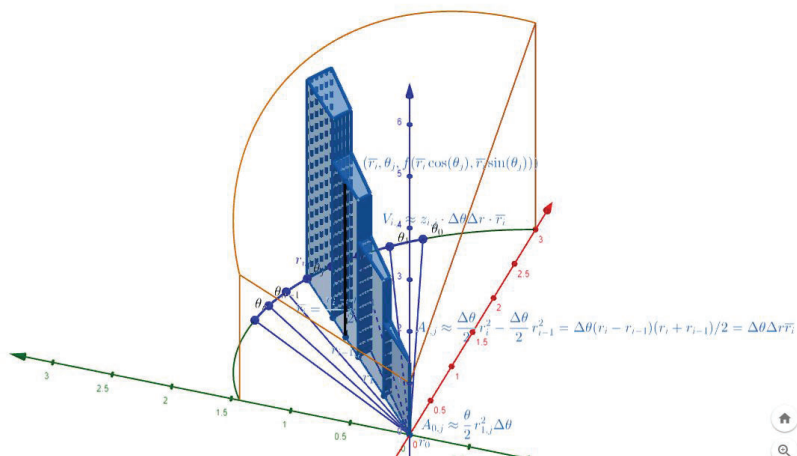


Figure 16.3.1: Volume via Slicing in Polar Coordinates

View in Geogebra: <https://www.geogebra.org/3d/mpge2k8c>

Looking down on the region in the  $xy$ -plane from above we can see the slices every  $\Delta\theta$  so that the interval  $\alpha \leq \theta \leq \beta$  is cut up into subintervals  $[\alpha = \theta_0, \theta_1], [\theta_1, \theta_2], \dots, [\theta_{j-1}, \theta_j], \dots, [\theta_{n-1}, \theta_n]$  where  $n$  is the number of slices over the  $\theta$ -interval. We also see the slices every  $\Delta r$  so that the interval  $r_0 \leq r \leq r_f$  is cut up into subintervals  $[r_0, r_1], [r_1, r_2], \dots, [r_{i-1}, r_i], \dots, [r_{m-1}, r_m]$  where  $m$  is the number of slices over the  $r$ -interval.

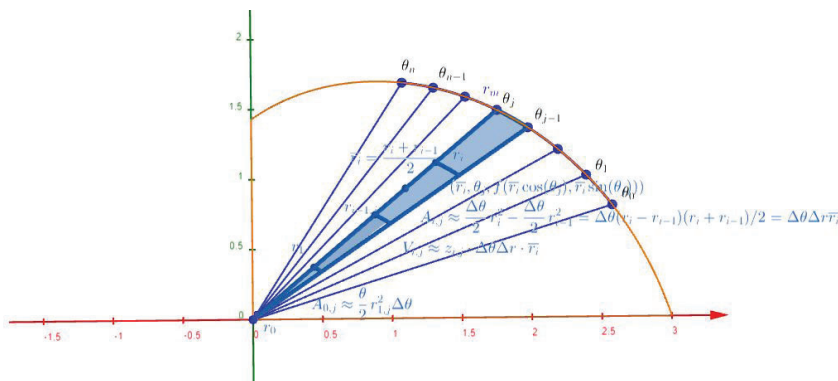


Figure 16.3.2: Slicing a Region in Polar Coordinates

View in Geogebra: <https://www.geogebra.org/3d/mpge2k8c>

Let's highlight one of the slices to determine a formula for its area. In figure 16.3.3 below, we can calculate that the area of the highlighted region,  $A_{i,j}$  is approximately

$$\begin{aligned} A_{i,j} &\approx \frac{\Delta\theta}{2} r_i^2 - \frac{\Delta\theta}{2} r_{i-1}^2 \\ &= \frac{\Delta\theta}{2} (r_i - r_{i-1})(r_i + r_{i-1}) \\ &= \Delta\theta \Delta r \cdot \frac{r_i + r_{i-1}}{2} \\ &= \Delta\theta \Delta r \cdot \bar{r}_i \end{aligned}$$

where  $\bar{r}_i = \frac{r_i + r_{i-1}}{2}$  is the average of  $r_i$  and  $r_{i-1}$ . The formula  $\frac{\Delta\theta}{2} r_i^2$  gives the area of a sector swept out by  $\Delta\theta$  and can be remembered as a fraction of the area of a circle.

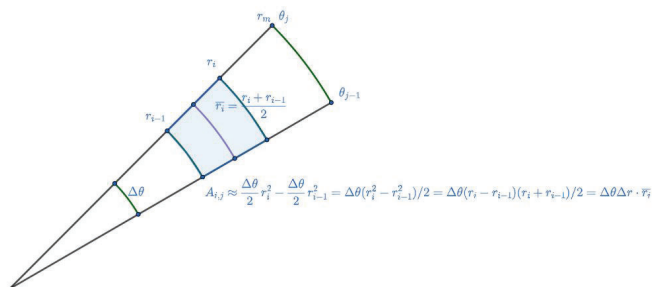


Figure 16.3.3: Slicing a Region in Polar Coordinates

View in Geogebra: <https://www.geogebra.org/graphing/qffh4fkb>

Now that we have the area of a slice,  $A_{i,j} \approx \Delta\theta\Delta r \cdot \bar{r}_i$ , we now have the volume of a prism being

$$V_{i,j} \approx z_{i,j} \cdot A_{i,j} = f(x_i, y_i) \cdot \Delta\theta\Delta r\bar{r}_i.$$

By substituting  $x_i = \bar{r}_i \cos(\theta_j)$  and  $y_i = \bar{r}_i \sin(\theta_j)$  we can get this volume in terms of just  $r_i$  and  $\theta_j$  to be

$$V_{i,j} \approx f(\bar{r}_i \cos(\theta_j), \bar{r}_i \sin(\theta_j)) \cdot \bar{r}_i \Delta\theta\Delta r.$$

Finally, we can add all of the volumes of the prisms up, and then let the number of slices  $m$  and  $n$  over the  $\theta$  and  $r$ -intervals respectively go to infinity to get our exact volume

$$V = \lim_{m,n \rightarrow \infty} \sum_{j=1}^n \sum_{i=1}^m f(\bar{r}_i \cos(\theta_j), \bar{r}_i \sin(\theta_j)) \cdot \bar{r}_i \Delta r \Delta \theta = \int_{\alpha}^{\beta} \int_{r_0}^{r_f} f(r \cos(\theta), r \sin(\theta)) \cdot r \, dr \, d\theta.$$

## 16.4 Volumes of Solids

**Example 16.4.1** Find the volume of the solid that is enclosed by the hyperboloid  $-x^2 - y^2 + z^2 = 1$  and the plane  $z = 2$ .

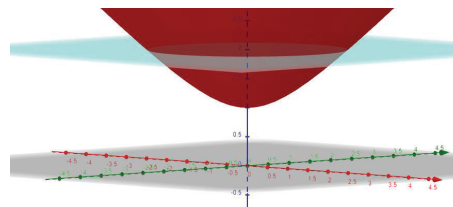
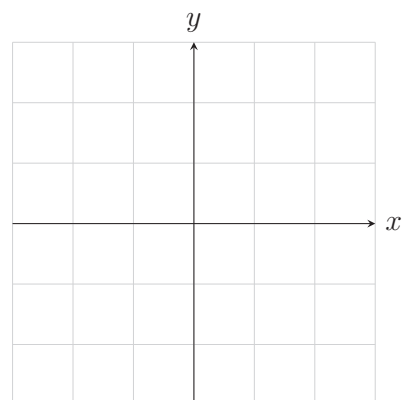


Figure 16.4.1: Volume between  $-x^2 - y^2 + z^2 = 1$  and  $z = 2$

View in Geogebra:

<https://www.geogebra.org/3d/g8wzcr4j>



**Exercise 16.4.1** Find the volume of the solid that lies below  $f(x, y) = x + y^2$  to the left of the  $y$ -axis and between the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .

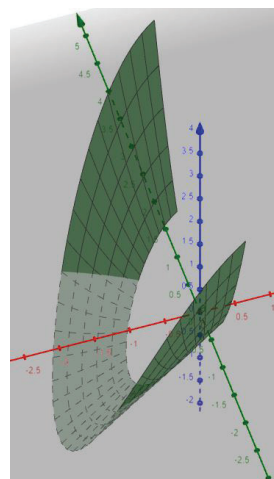
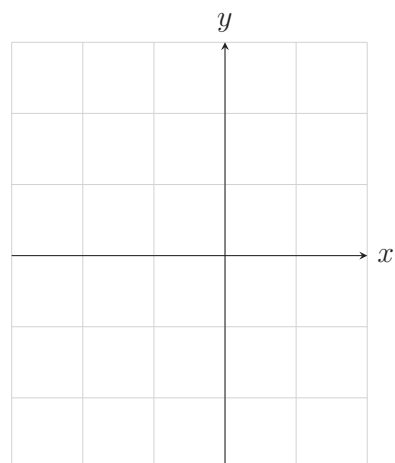


Figure 16.4.2: Volume below  $f(x, y) = x + y^2$  to the left of the  $y$ -axis and between the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$

View in Geogebra:

<https://www.geogebra.org/3d/jxxzrreg>



## 16.5 Polar Regions whose Radius Varies with Angle

**Example 16.5.1** Find the volume under  $f(x, y) = x + y + 1$  and inside the region bounded by the first quadrant and the spiral  $r = 3 - \theta$ .

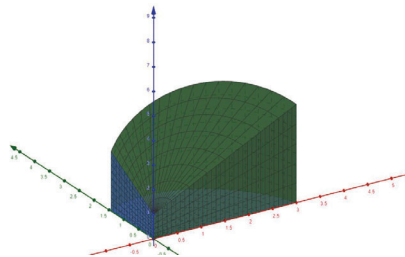


Figure 16.5.1: volume under  $f(x, y) = x + y + 1$  and inside the region bounded by the first quadrant and the spiral  $r = 3 - \theta$

View in Geogebra:

<https://www.geogebra.org/3d/ppgdnr7j>

