LESSON 13

Double Integrals over Rectangles

Contents

13.1 Definite Integral Review

In single-variable integral calculus, the definite integral is defined to be the area between a curve and the horizontal axis. Given a function, $y = f(x)$, the area from $x = a$ to $x = b$ under the curve can be found by cutting up the interval $[a, b]$ into n subintervals, constructing a rectangle for each subinterval whose width is that of the subinterval and whose height is some y-value of the curve within the subinterval. We then sum the areas of each of these rectangles to obtain an approximation of the area under the curve. Then, to obtain the exact area under the curve, we take the limit as n goes to infinity so that we have constructed an infinite number of rectangles of infinitesimal width, and heights that match every y-value of the curve so that the sum of their areas is the exact area under the curve. This can be done by choosing the widths of the subintervals to be all different and to choose the heights of the rectangles to be any y-value within the subinterval on the curve, and moreover could be done with trapezoids instead of rectangles. Thus there is lots of flexibility in how we construct finding the area. However, a simple way of constructing this is to cut the interval $[a, b]$ into n subintervals, $[a = x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n]$ of equal width $\Delta x = \frac{b-a}{n}$ so we can thus find that $x_i = x_0 + \Delta x \cdot i$ for $i = 0, 1, 2, ..., n$. We then choose the height of the rectangles for each subinterval to be that of the y-value on the curve that is on the left endpoint of the interval so that $h_i = f(x_i)$ are the heights of rectangles 0 through $n - 1$. Thus, the area under the curve can in this way be found to be:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x,$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 0, 1, ..., n-1$.

Example 13.1.1 Find the area under the curve $f(x) = \frac{1}{2}x + 3$ over the interval $[0, 4]$ via the Riemann Sum as described above.

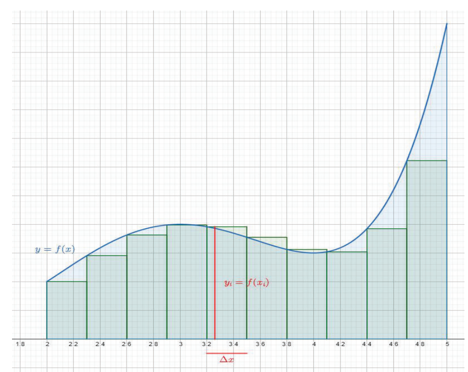

Figure 13.1.1: Finding the Area Under a Curve Using Left-Rectangles.
View in Geogebra: https://www.geogebra.org/graphing/fvcncg9e

13.2 Volume, Double Integrals, and Riemann Sums

We may now extend this idea to functions of two variables to find the volume between the xy-plane and a surface given by $z = f(x, y)$.

Let Region $R = [a, b] \times [c, d]$ represent $\{(x, y) \,|\, a \le x \le b, c \le y \le d\}$. Define $f(x, y)$ on R and, for the moment, assume $f(x, y) \ge 0$. Now subdivide $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ and $[c, d]$ into m subintervals $[y_0, y_1], [y_1, y_2], \ldots, [y_{m-1}, y_m]$ to obtain rectangles $r_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for $i = 0, 1, 2, ..., n-1$ and $j = 0, 1, 2, ..., m-1$.

Note that we could construct the sub-rectangles to not be of uniform widths and still have this work out, but for simplicity we will construct them to all have equal widths

$$\Delta x = \frac{b-a}{n}$$

and equal lengths

$$\Delta y = \frac{d-c}{m}.$$

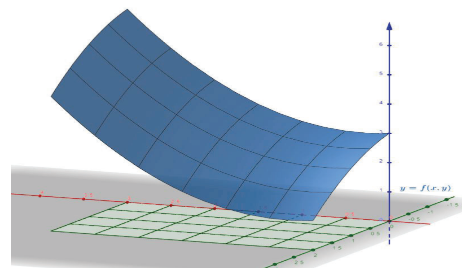

Figure 13.2.1: Finding the Area Under a Curve Using Rectangles.
View in Geogebra:
https://www.geogebra.org/3d/bzc449g4

We can then inscribe a rectangular solid on each sub-rectangle whose widths and lengths are all Δx and Δy respectively, and whose heights are $h_{i,j} = f(x_i^*, y_j^*)$ where (x_i^*, y_j^*) is any point in $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. Thus the volume of rectangular solid $S_{i,j}$ is $V_{i,j} = f(x_i^*, y_j^*)\Delta x \Delta y$.

For samples points x_i^* and y_j^*, we define the **double integral** over the region R to be the volume of the solid between R in the xy-plane, and $f(x, y)$:

$$\begin{aligned} V &= \iint_R f(x, y)\, dA \\ &= \lim_{m,n\to\infty} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(x_i^*, y_j^*)\Delta x \Delta y \end{aligned}$$

where dA is the infinitesimal area of the rectangles $dxdy$.

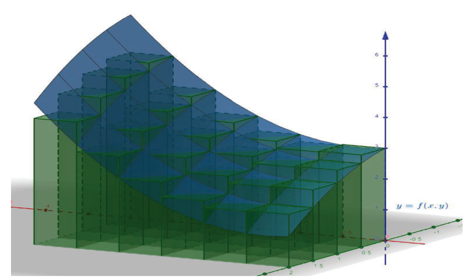

Figure 13.2.2: Finding the Area Under a Curve Using Rectangles.
View in Geogebra:
https://www.geogebra.org/3d/bzc449g4

Later in this lesson we will look at choosing (x_i^*, y_j^*) to be the midpoints of the rectangles $r_{i,j}$, but for now we will choose (x_i^*, y_j^*) to be the lower-left points of each rectangle so that $x_i^* = x_i = x_0 + \Delta x i$ and $y_j^* = y_j = y_0 + \Delta y j$ for $i = 0, 1, 2, ..., n-1$ and $j = 0, 1, 2, ..., m-1$.

Example 13.2.1 Find $\displaystyle\iint_R 4\,dA$ for $R = \{(x, y)\,|-2 \le x \le 2, 1 \le y \le 6\}$

Example 13.2.2 Find $\displaystyle\iint_R (x + 2y^2)\,dA$ for $R = [0, 2] \times [0, 4]$ using upper-right corners for sample points with $\Delta x = 0.5$ and $\Delta y = 1$.

Example 13.2.3 Find $\iint\limits_R (x^2 - 2xy + 3y)dA$ for $R = [0, 1] \times [0, 3]$ using **upper-left** corners for sample points with $\Delta x = 0.25$ and $\Delta y = 0.5$.

Exercise 13.2.1 Find $\iint\limits_R (3x - 2xy)dA$ for $R = [0, 2] \times [0, 3]$ exactly using **lower-left** corners for sample points.

Example 13.2.4 A volume of sediment is being measured in a geological survey. The depth of the sediment is measured every three meters in both directions and is given in meters in the following table.

$x \backslash y$	0	3	6	9	12
0	0	2	4	3	0
3	0	3	5	4	1
6	1	4	6	5	2
9	1	2	4	3	1
12	0	1	3	2	1

Use upper-left-corners to approximate the volume of the sediment.

13.3 Midpoint Rule

As seen in the previous examples and exercise, we can formulate our Riemann Sum to calculate volume using lower-left corners, lower-right corners, upper-left corners, or upper-right corners of the rectangles. Since we are taking the limit as m and n go to infinity, it makes our rectangles infinitesimals so the y-values of the corners collapse on each other. Similarly we may reformulate the Riemann Sum again using the midpoints of the rectangles.

Definition 13.3.1

Given $z = f(x, y)$, the volume between the surface and the xy-plane over the region $R = [a, b] \times [c, d]$ where $[a, b]$ is divided into n subintervals and $[c, d]$ is divided into m subintervals is given by

$$V = \iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(\bar{x}_i, \bar{y}_j)\Delta x \Delta y$$

where $\bar{x}_i$ is the midpoint between $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint between $[y_{j-1}, y_j]$. This method is known as the **Midpoint Rule**.

Supposing, as with the corner formulations, that we divide $[a, b]$ into n equal-width sub-invervals and $[c, d]$ into m equal-width subintervals, then

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad \Delta y = \frac{d-c}{m}$$

with

$$\bar{x_i} = \frac{x_0 + x_1}{2} + \Delta x(i-1) \quad \text{and} \quad \bar{y_j} = \frac{y_0 + y_1}{2} + \Delta y(j-1).$$

Exercise 13.3.1 Approximate $\iint\limits_R (2x + 3y^2)dA$ for $R = [0, 2] \times [0, 3]$ with $n = 4$ and $m = 3$, using midpoints for sample points.

13.4 Average Value

Let's build up the idea of average value for the volume between the xy-plane and a surface given by $z = f(x, y)$ on some given domain, D.

First, let's look at the classic computation for the average of some set of numbers. Given some set of y-values, $\{y_1, y_2, ..., y_n\}$, then the average value is

$$y_{ave} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

If we rearrange this equation we get

$$n \cdot y_{ave} = \sum_{i=1}^{n} y_i.$$

Writing it this way is to highlight that the average value is a height (y-value) such that multiplying it by the total number of y-values gives us the same as the total of all of the y-values.

This idea can then be extended the the average-value of a continuous function over some interval $[a, b]$ so that

$$(b-a)\cdot y_{ave} = \int_a^b y(x)dx.$$

We can again see that the average y-value is the average height so that the width times this y_{ave} gives us the area under the curve.

We now extend this to the idea of the average z-value of some function $z = f(x, y)$ over some domain, D, so that

$$A(D)\cdot z_{ave} = \iint_D f(x, y)dA$$

where $A(D)$ is the area of domain.

Thus we can compute the average z-value via the formula

$$z_{ave} = \frac{1}{A(D)}\iint_D f(x, y)dA.$$

Example 13.4.1 The contour map shows the temperature, in degrees Fahrenheit, at 4:00 PM on February 26, 2007 in Colorado. (The state measures 388 miles west to east and 276 miles south to north.) Use the Midpoint Rule with $m = n = 4$ to estimate the average temperature in Colorado at that time.

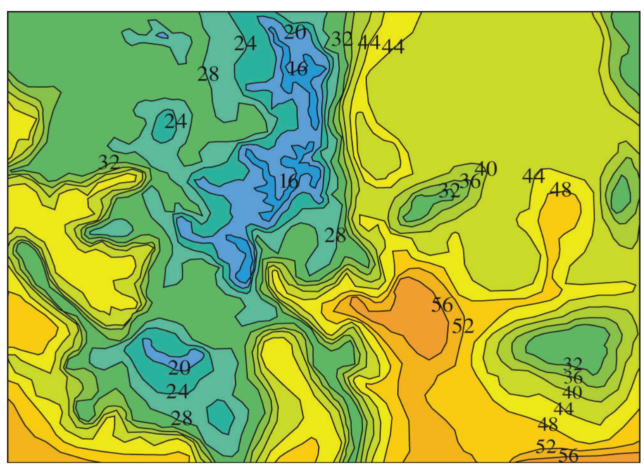

Figure 13.4.1: Contour Temperature Map