LESSON 12

Maximum and Minimum Values

Contents

12.1 Review of Minimums and Maximums in Single-Variable Calculus

Recall from single-variable calculus the definition of **local maximums** and **local minimums**:

Definition 12.1.1

Given a function $y = f(x)$, with domain D_f, $y_1 = f(x_1)$ is a **local maximum** if there exists an open interval, $(a, b) \in D_f$ with $a < x_1 < b$ such that $y_1 \geq f(x)$ for all $x \in (a, b)$. Moreover, $y_2 = f(x_2)$ is a **local minimum** if there exists an open interval, $(c, d) \in D_f$ with $c < x_2 < d$ such that $y_2 \leq f(x)$ for all $x \in (c, d)$.

Essentially this is saying that a point on the graph of f has a y-value that is a local maximum if there are points immediately to both sides of it whose y-values are less than *or equal to* the y-value in question. And, f has a y-value that is a local minimum if there are points immediately to both sides of it whose y-values are greater than *or equal to* the y-value in question. This excludes endpoints from having their y-values be *local* maximums or minimums.

We can also talk about **absolute maximums and minimums** which have the following definitions:

Definition 12.1.2

Given a function $y = f(x)$ with domain, D_f, y_1 is ***the* absolute maximum** if $y_1 \geq f(x)$ for *all* $x \in D_f$. Similarly, y_2 is ***the* absolute minimum** if $y_2 \leq f(x)$ for *all* $x \in D_f$.

Note that the absolute maximum and absolute minimum may occur endpoints.

Example 12.1.1 Take the function $y = f(x)$ shown in figure 12.1.1 below. What are the local and absolute maximums and minimums and where do they occur?

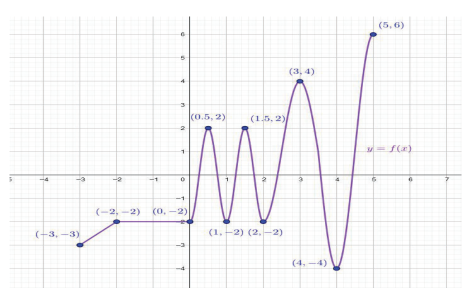

Figure 12.1.1: Graph of a function $y = f(x)$.
View in Geogebra:
https://www.geogebra.org/graphing/hna8ua5d

12.1.1 Critical Points of Functions in One Variable

As we review local maximum and minimum values of functions in one-variable, let us note again that these values can only occur when their corresponding input-value (usually x) is a **critical number**.

Definition 12.1.3

Given a function $y = f(x)$ with domain D_f, then $x_1 \in D_f$ is a critical number of f if either $f'(x_1) = 0$ *or* $f'(x_1)$ is undefined.

Now, not every critical number will yield a local maximum or minimum when plugged into the function, but these critical numbers are the only places that we *could* get a local maximum or minimum since local maximums and minimums occur with either a slope of zero or at a sharp point.

Now, the second-derivative test will allow us to determine if a critical number in the domain of f' is a local minimum or maximum and this test will be analogous to what we are going to do in functions in two-variables.

Theorem 12.1.1

Supposing x_1 is a critical number of a function f and x_1 is in the domain of $f'(x)$ and $f''(x)$ (so we are not dealing with a sharp point on f or f'), then:

- If $f''(x_1) > 0$, $f(x_1)$ is a local minimum.
- If $f''(x_1) < 0$, $f(x_1)$ is a local maximum.
- If $f''(x_1) = 0$, $(x_1, f(x_1))$ is a saddle point.

Absolute maximums and minimums can only occur at either a local maximum or minimum or at an endpoint!

Example 12.1.2 Given $f(x) = 2x^3 - 15x^2 + 24x + 7$, find the absolute minimum and absolute maximum on $[0, 6]$.

12.2 Local Minimum and Maximum Values

Local minimums and maximums of functions in two-variables are analogous to local minimums and maximums of functions in one-variable. A local maximum is a z-value on the graph of the function that is the highest z-value within some localized region of the graph and a local minimum is similarly a z-value which is the lowest within some localized region.

Definition 12.2.1

Given a function $z = f(x, y)$ with domain D_f, $z_1 = f(x_1, y_1)$ is a **local maximum** if there exists an open disc, $D \in D_f$ with $(x_1, y_1) \in D$ such that $z_1 \geq f(x, y)$ for all $(x, y) \in D$. Moreover, $z_2 = f(x_2, y_2)$ is a **local minimum** if there exists an open disc, $D \in D_f$ with $(x_2, y_2) \in D$ such that $z_2 \leq f(x, y)$ for all $(x, y) \in D$.

Note, since these local extrema (maximums and minimums) have to occur on an open disc, they cannot occur on an edge of a closed domain. This is analogous to not having local extrema occur at an endpoint of a function in a single-variable.

Given a function in a single-variable we have what's called a **saddle point**, which is mentioned in theorem 12.1.1. This is where the function is concave up on one side of the point and concave down on the other.

When dealing with function in two-variables, the same thing can occur where we could get a "flat" spot on the graph (where the tangent plane is a constant z-value so it's slope in all directions is 0) that is neither a local maximum nor a local minimum. As with our single-variable situation, this occurs where the function slopes down from this point in some directions, and slopes up from this point in other directions.

Example 12.2.1 Given the function $z = f(x, y)$ given in the graph shown below, determine the local extrema and the saddle points.

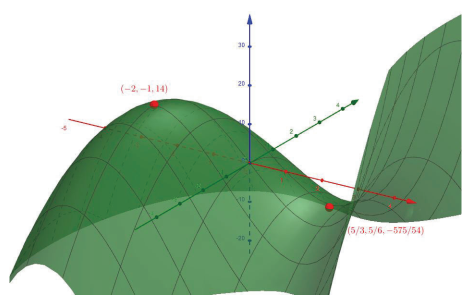

Figure 12.2.1: Graph of a function $z = f(x, y)$.
View in Geogebra:
https://www.geogebra.org/3d/rm2uejyq

12.3 Critical Points of Functions in Two Variables

Before we look at how to tell if a critical point in a two-variable function will yield a local minimum, local maximum, or saddle point, let's review the Taylor series expansion for a function which is smooth and continuous near $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + ...$$

This is for x-values near $x = a$. Thus if we want to move by h, we get

$$f(x+h) \approx f(a) + f'(a)h + \frac{f''(a)}{2}h$$

by using the second-order Taylor approximation. Similarly, given a function in two-variables, if we move by h in the x-direction and k in the y-direction from a point (a, b), we use the directional derivative with $\mathbf{u} = \langle h, k\rangle$ to get the second-order Taylor approximation

$$\begin{aligned} f(a+h, b+k) &\approx f(a,b) + D_{\mathbf{u}}f(a,b) + \frac{D^2_{\mathbf{u}}f(a,b)}{2} \\ &= f(a,b) + hf_x(a,b) + kf_y(a,b) + \frac{h^2}{2}f_{xx}(a,b) + hkf_{xy}(a,b) + \frac{k^2}{2}f_{yy}(a,b) \end{aligned}$$

which we see is analogous to $f(x+h)$ for a single-variable function noted above.

Now, a critical point of a function $f(x, y)$ is defined to be when $f_x(a, b) = f_y(a, b) = 0$. This if (a, b) is a critical point of f, we may plug this into the above second-order approximation we get

$$\Delta z = f(a+h, b+k) - f(a,b) \approx \frac{h^2}{2}f_{xx} + hkf_{xy} + \frac{k^2}{2}f_{yy}$$

We may now classify critical points in the following manner:

For h, k small enough, $f(a+h, b+k) - f(a, b) = \frac{h^2}{2}(f_{xx} + 2\frac{k}{h}f_{xy} + \frac{k^2}{h^2}f_{yy})$.

Notice this has characteristics of 'quadratic in form.' For example, with $Q(w) = Aw^2 + 2Bw + C$, which changes signs when it has two distinct roots when $B^2 - AC > 0$ and doesn't change signs when $B^2 - AC \leq 0$. The quadratic form we have has k/h is its variable and f_{yy}, f_{xy}, and f_{xx} as the coefficients. Thus Δz has two distinct roots when $(f_{xy})^2 - f_{xx}f_{yy} > 0$ and doesn't change signs when $(f_{xy})^2 - f_{xx}f_{yy} \leq 0$.

So, if $f(a+h, b+k) - f(a, b)$ *changes sign*, then in one direction f is increasing and in the other direction f is decreasing (i.e., a *saddle point*). This happens when $(f_{xy})^2 - f_{xx}f_{yy} > 0$.

Also, $(f_{xy})^2 - f_{xx}f_{yy} \leq 0$ when $f_{xx} > 0$ and $f_{yy} > 0$ implies a local minimum. Likewise, $(f_{xy})^2 - f_{xx}f_{yy} \leq 0$ when $f_{xx} < 0$ and $f_{yy} < 0$ implies a local maximum.

To summarize,

- Saddle Point: $(f_{xy})^2 - f_{xx}f_{yy} > 0$
- Local Min: $(f_{xy})^2 - f_{xx}f_{yy} \leq 0$ for $f_{xx} > 0$ and $f_{yy} > 0$
- Local Max: $(f_{xy})^2 - f_{xx}f_{yy} \leq 0$ for $f_{xx} < 0$ and $f_{yy} < 0$

Notice that if $f_{xx} = f_{xy} = f_{yy} = 0$, a different approach may need to be considered.

Example 12.3.1 Find and classify the critical points of $f(x, y) = x^2 - y^2$.

Exercise 12.3.1 Find and classify the stationary points of $f(x, y) = x^3 - y^3 - 3xy + 4$.

12.4 Absolute Minimum and Maximum

Again, as with single-variable functions, the **absolute maximum** and **absolute minimum** of a two-variable function are the largest and smallest outputs of the function on its entire domain.

Definition 12.4.1

Given a function $z = f(x, y)$ with domain D, then z_1 is the **absolute maximum** if $z_1 \geq f(x, y)$ for all $(x, y) \in D$ and z_2 is the **absolute minimums** if $z_2 \leq f(x, y)$ for all $(x, y) \in D$.

Now, in two-variable functions, we could have a closed domain with an edge along its border that could potentially contain an absolute maximum or minimum. Thus, to find the absolute maximum or minimum of a function in two-variables with a closed domain, we must look at

- critical values inside the closed domain,
- critical values on the edge of the domain,
- the outputs at any corners of the edge of the domain.

Example 12.4.1 Find the absolute maximum and absolute minimum of $f(x, y) = 3 + xy - x - 2y$ on closed triangular region D with vertices $(1, 0)$, $(5, 0)$, and $(1, 4)$.

Exercise 12.4.1 Find the absolute maximum and absolute minimum of $f(x, y) = x^3 - 3x - y^3 + 12y$ on the closed rectangular region D with vertices $(-2, 3)$, $(2, 3)$, $(2, -2)$, and $(-2, -2)$.

Example 12.4.2 The base of an aquarium with given volume V has a slate bottom and glass sides. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimizes the cost of the materials.