LESSON 9

Tangent Planes and Linear Approximations

Contents

In this lesson we will be looking at finding equations of tangent planes for functions in two variables along with linear approximations for Δx, Δy, and Δz at a given point, and the instantaneous changes in x, y, and z called the differentials dx, dy, and dz.

9.1 Tangent Planes

To begin let us note that the equation of a plane is

$$ax + by + cz = d$$

where $\mathbf{n} = \langle a, b, c \rangle$ is a normal vector for the plane and d is found by taking a point $P = (x_0, y_0, z_0)$ on the plane and plugging it in after having found a, b, and c.

Example 9.1.1 Suppose $P = (1, 2, -2)$ is a point on a plane with normal vector $\mathbf{n} = \langle -3, 5, 2 \rangle$. Determine the equation for the plane.

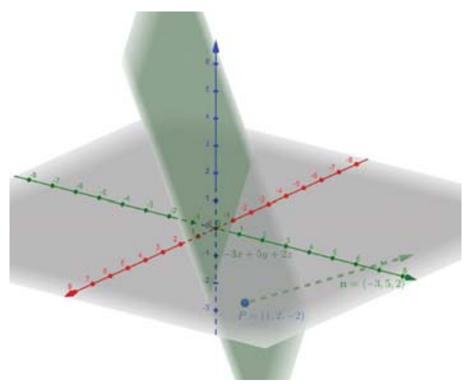

Figure 9.1.1: Plane with point $P = (1, 2, -2)$ and normal vector $\mathbf{n} = \langle -3, 5, 2 \rangle$
View in Geogebra:
https://www.geogebra.org/3d/cx6bqfr5

Suppose now that we have a function $z = f(x, y)$ and we are looking for the equation of the tangent plane at a point (x_0, y_0) where the function is differentiable. To find the equation of this plane we need a point on the plane and its normal vector.

The point we want is $P = (x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$.

The normal vector will be acquired by taking the cross product of the tangent vectors at the point in the x and y-directions respectively.

From our previous lesson we have

$$f_x(x, y) = \frac{\partial z}{\partial x}$$

is the slope of the tangent line in the x-direction. Thus the direction of the tangent line through our point in the x-direction is $\langle \partial x, 0, \partial z \rangle = \langle 1, 0, f_x \rangle$. Similarly

$$f_y(x, y) = \frac{\partial z}{\partial y}$$

is the slope of the tangent line in the y-direction. So the direction of the tangent line through our point in the y-direction is $\langle 0, \partial y, \partial z \rangle = \langle 0, 1, f_y \rangle$.

Now both of these vectors, $\langle 1, 0, f_x \rangle$ and $\langle 0, 1, f_y \rangle$ lie in our tangent plane and are not parallel. Thus, taking their cross product will give us the normal vector which is perpendicular to both.

$$\mathbf{n} = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \langle -f_x, -f_y, 1 \rangle.$$

Thus we can begin to construct the equation of our tangent plane by writing

$$-f_x x - f_y y + z = d.$$

We then plug in our point (x_0, y_0, z_0) to get

$$d = -f_x x_0 - f_y y_0 + z_0.$$

Putting this together we get

$$-f_x x - f_y y + z = -f_x x_0 - f_y y_0 + z_0$$

which can be rearranged by solving for z to get

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$$

as the equation of our tangent plane.

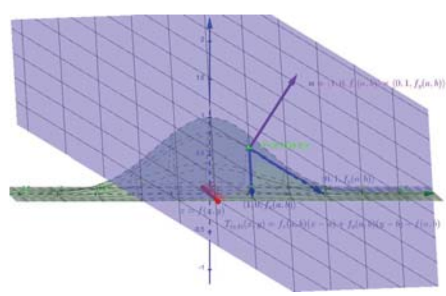

Figure 9.1.2: Tangent Plane
View in Geogebra:
https://www.geogebra.org/3d/wyykz53c

Theorem 9.1.1

The equation of the tangent plane to a function f at (x_0, y_0) is

$$T_{(x_0,y_0)}(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

Example 9.1.2 Find the tangent plane for the paraboloid $f(x,y) = 3 - x^2 - 3y^2$ at the point when $(x,y) = (2,1)$ in both standard form $[ax + by + cz = d]$ and linearized form $[T_{(x_0,y_0)}(x,y) = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + f(x_0,y_0)]$.

Exercise 9.1.1 Find the tangent plane for the function $f(x,y) = x\sqrt{y}$ at the point when $(x,y) = (1,4)$.

Exercise 9.1.2 Find the tangent plane for the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

when $(x,y) = (0,0)$.

9.2 Linear Approximations

A tangent plane to a function in two variables at a given point *is* the linear approximation to the function near the point. Say a function $z = f(x, y)$ is linearized around the input (x_0, y_0) using the tangent plane $L_{(x_0,y_0)}(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$. Then for points near (x_0, y_0), $L_{(x_0,y_0)}(x, y)$ will give us outputs approximately equal to the outputs of $f(x, y)$. The error of the linearization will be dependent on the function and how quickly its z-values are changing.

Say we move by $(\Delta x, \Delta y)$ from (x_0, y_0) which has the z-value $z_0 = f(x_0, y_0)$. Then our new z-value is $z_1 = f(x_0 + \Delta x, y_0 + \Delta y)$. Then

$$\Delta z = z_1 - z_0 = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

If we use the linearization we get

$$z_1 \approx f_x(x_0, y_0)((x_0+\Delta x)-x_0)+f_y(x_0, y_0)((y_0+\Delta y)-y_0)+z_0 = f_x(x_0, y_0)\Delta x+f_y(x_0, y_0)\Delta y+z_0.$$

Then using the linearization our change in z is

$$\Delta z = z_1 - z_0 \approx (f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + z_0) - z_0 = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

Instead of using the approximate symbol we write

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

so that ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$.

Example 9.2.1 Given $f(x, y) = xe^{xy}$, find the linearization at $(x, y) = (1, 0)$. Compare the outputs of $f(1.1, -0.1)$ and $L_{(1,0)}(1.1, -0.1)$. What is the actual Δz between $f(1, 0)$ and $f(1.1, -0.1)$? What is the approximate Δz that we get using the linearization?

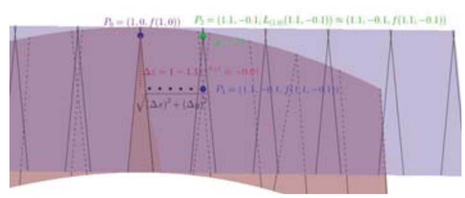

Figure 9.2.1: Linear Approximation for $f(x, y) = xe^{xy}$ near $(x, y) = (1, 0)$
View in Geogebra:
https://www.geogebra.org/3d/j74zcpb8

While the true power of linearization becomes most apparent when you get into math classes that combine linear algebra with dynamical systems, we can look at the important application of it for when we only have a set of data for a function as opposed to an explicitly defined symbolic function.

Example 9.2.2 Suppose $z = f(x, y)$ is defined via the following table.

x \ y	6	9	12	15	18
8	12	15	17	16	16
10	17	21	24	24	23
12	21	25	28	30	31
14	24	25	29	33	35
16	26	27	28	28	30

Find a linear approximation for $z = f(x, y)$ when x is near 14 and y is near 9. Use it to approximate $f(14.5, 8.8)$.

9.3 Differentials

Given a function $y = f(x)$, then we know that $\dfrac{dy}{dx} = f'(x)$. In this case we define the **differentials**, dy and dx as:

Definition 9.3.1

Given a differentiable function $y = f(x)$, the **differentials** are

$$dx \text{ is the change in } x$$

and

$$dy = f'(x)dx \text{ is the change in the } y-\text{value of the tangent line.}$$

While $dx = \Delta x$ is the exact change in x, we find that $dy \approx \Delta y$ are not equal unless f is linear over the interval Δx since Δy is the actual change in the output of the function over Δx while dy is the change in the output of the tangent line.

Similarly, given a function $z = f(x, y)$ we define the **differential** dz (or **total differential** to be:

Definition 9.3.2

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Exercise 9.3.1 Given the function $m = f(p, q) = p^5 q^3$, if p changes from 2 to 2.05 and q changes from 3 to 2.96, compare the values of Δm and dm.

Example 9.3.1 The dimensions of a closed rectangular box are measured as 80cm, 60cm, and 50cm respectively with a possible error of 0.2cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

9.4 Tangent Planes to Parametric Surfaces

Suppose we have a parametric surface

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

which we want to linearize with a tangent plane at the point $(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))$.

Remember that by holding u or v constant we travel along the gridlines of the surface given by $\mathbf{r}$. Thus if we take partial derivatives of $\mathbf{r}$ with respect to u and v we will tangent vectors in the u and v directions respectively. We define the partial derivatives of $\mathbf{r}(u,v)$ to be

$$\mathbf{r}_u(u,v) = \langle \frac{\partial x}{\partial u}(u,v), \frac{\partial y}{\partial u}(u,v), \frac{\partial z}{\partial u}(u,v)\rangle$$

and

$$\mathbf{r}_v(u,v) = \langle \frac{\partial x}{\partial v}(u,v), \frac{\partial y}{\partial v}(u,v), \frac{\partial z}{\partial v}(u,v)\rangle$$

We can therefore find the normal vector to the tangent plane at $\mathbf{r}(u_0,v_0)$ by taking the cross product of $\mathbf{r}_u(u_0,v_0)$ and $\mathbf{r}_v(u_0,v_0)$. That is

$$\mathbf{n}_{(u_0,v_0)} = \mathbf{r}_u(u_0,v_0) \times \mathbf{r}_v(u_0,v_0).$$

Example 9.4.1 Find the tangent plane to the parametric surface $x = u^2$, $y = v^2$, $z = uv$ when $u = 1$ and $v = 1$.

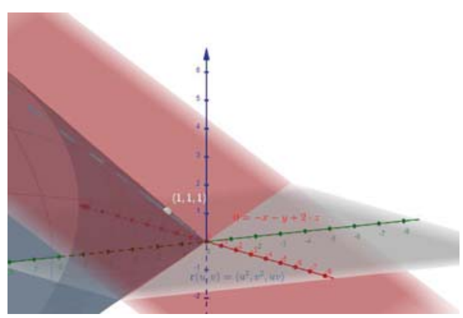

Figure 9.4.1: Tangent Plane for $\mathbf{r}(u,v) = \langle u^2, v^2, uv\rangle$ when $u = 1$ and $v = 1$
View in Geogebra:
https://www.geogebra.org/3d/tugd26bx