LESSON 8

Partial Derivatives

Contents

8.1 Introduction and Definitions

Our goal over the next two lessons is to come up with a linear approximation to a function in two variables at a given point. If we think about a function in two variables graphing as a landscape, then a linearization of the surface will be a tangent plane as opposed to a tangent line. We do not now have a single slope but different slopes depending on the direction being traveled.

Look at figure 8.1 below. We see that if you travel along the surface in the x-direction near point A you will be on the red tangent line labeled $T_x(a, b)$. But if you travel along the surface in the y-direction near this point you will be on the green tangent line labeled $T_y(a, b)$.

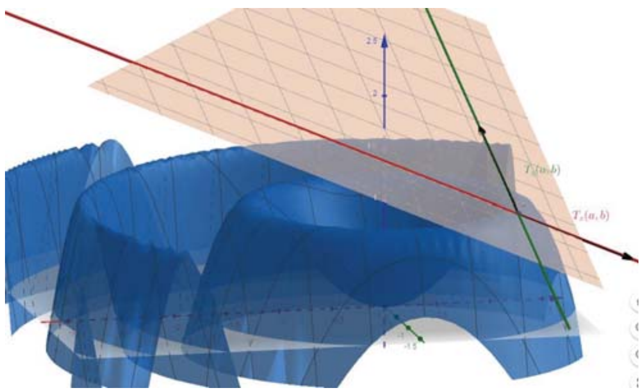

Figure 8.1.1: Graph of a surface with a tangent plane and tangent lines in the x and y-directions
View in Geogebra: https://www.geogebra.org/3d/bvv8jqjp

In order to find the equation of the tangent lines and tangent plane we need the slopes of the tangent lines in the x and y directions at the point $A = (a, b)$. Let's begin with the following definition:

Definition 8.1.1

Given a function, f, in two variables then

$$f_x(a, b) = \lim_{x \to a} \frac{f(x, b) - f(a, b)}{x - a}$$

is the slope of f when traveling in the x-direction and

$$f_y(a, b) = \lim_{y \to b} \frac{f(a, y) - f(a, b)}{y - b}$$

is the slope of f when traveling in the y-direction.

This definition is true and comes directly from our definition of a derivative in one-variable. However, we are going to find **partial derivative** functions that allow us to find the slopes of f when traveling in the x and y-directions at any point in the domain of f (that has a defined slope in the given direction) respectively. We define these functions here:

Definition 8.1.2

Given a function, f, in two variables then

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

is a new function in x and y called the **partial derivative of f with respect to x** and outputs the slopes of f in the x-direction for any given point in its domain. Similarly,

$$\frac{\partial}{\partial y} f(x,y) = f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

is a new function in x and y called the **partial derivative of f with respect to y** and outputs the slopes of f in the y-direction for any given point in its domain.

8.2 Computing the Partial Derivatives

Again, the definitions in Definition 8.1.2 are true and again come from the definition of a derivative in one-variable but we are not going to use them in practice. To simplify things we find $f_x(x,y)$ by taking the derivative of f with respect to x while thinking of the y as a constant instead of a variable. Similarly to find $f_y(x,y)$ we will take the derivative of f with respect to y while thinking of the x as a constant instead of a variable.

Example 8.2.1 Given $f(x,y) = x^4 + x^2y - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$. Interpret these values as slopes.

Exercise 8.2.1 Given $f(x, y) = \arctan(y/x)$, find $f_x(2, 3)$ and $f_y(2, 3)$ and interpret them as slopes.

Definition 8.2.1

Suppose $z = f(x, y)$, then we define the following notations:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_x(x, y)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = f_y(x, y)$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

We will investigate the second partial derivatives defined above more in the coming lessons but for now we can think of f_{xx} as the concavity in the x-direction and f_{yy} as the concavity in the y-direction. To understand f_{xy} we think of it as the rate of change of f_x in the y-direction. That is, how do the slopes in the x-direction change as we move in the y-direction. Symmetrically f_{yx} is the rate of change of f_y in the x-direction - the rate of change of the slopes in the y-direction as we move in the x-direction. A handy theorem that works form most function that we run into is:

Theorem 8.2.1

Clairaut's Theorem Suppose a function f in two variables is defined on a disc, D. If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are both continuous on D, then

$$f_{xy}(x, y) = f_{yx}(x, y)$$

.

One more quick note on partial derivatives before doing more examples is that, without getting into a long and formal definition, if we want to take partial derivatives of higher order we can do so. For example, we could take

$$f_{xyy}(x, y) = \frac{\partial}{\partial y}(f_{xy}) = \frac{\partial^3 f}{\partial y^2 \partial x}.$$

Example 8.2.2 Find all second order partial derivatives for $f(x, y) = \sin(2x - 3y)$.

Exercise 8.2.2 Given $f(x, y) = 3xy^4 + x^3y^2$, find f_{xxy} and f_{yyy}.

8.3 Implicit Differentiation

Given an implicit equation in three variables it is possible to find partial derivatives with respect to x and y. To do this, we still think of z as a function in both x and y, when taking a partial with respect to x we still think of y as a constant and, symmetrically, if taking a partial with respect to y we still think of x as a constant.

Example 8.3.1 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $yz = \ln(x + z)$.

Exercise 8.3.1 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $\sin(xyz) = x + 2y + 3z$.