LESSON 7

Limits and Continuity

Contents

7.1 Limits in Two Variables

We want to investigate the concepts of limits and continuity of real-valued functions in two-variables. To do this we need a working definition of

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

for some function, f, in two-variables, some point, (a,b) in $\mathbb{R}^2$, and some real-number, L. Before jumping to the definition, let's remind ourselves of the definition of

$$\lim_{x\to a} f(x) = L$$

for a function, f, in one-variable, some real number, a, in $\mathbb{R}$, and some number, L, in $\mathbb{R}$:

Definition 7.1.1

Given a function, f, in one-variable, x, the **limit as x approaches some real number, a, equals L in $\mathbb{R}$**, that is

$$\lim_{x\to a} f(x) = L,$$

if we can get the outputs of f to become arbitrarily close to L by bringing x arbitrarily close to a (*from either side*) but not equal to a.

The key concept here is that we must approach a from both sides and have the limit be the same from both sides for this limit to exist. We can talk about one-sides limits, but the general limit must approach a from all (in this case only two since we are in one-dimension) directions.

To say that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that the outputs of f must approach L *from all directions*. But it is actually more than this. For the limit to equal L we need the outputs of f to approach L *along any path within the domain of f*. This means we could be following a linear path, a parabolic path, an exponential path or any path imaginable and all of these paths would need to output the same number for the limit to exist at that point. With that, let's take a look at the definition:

Definition 7.1.2

Given a function, f, in two-variables, x and y, the **limit as (x,y) approaches some point, $(a,b) \in \mathbb{R}^2$, equal L in $\mathbb{R}$**, that is

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

if we can get the outputs of f to become arbitrarily close to L by bringing (x,y) arbitrarily close to (a,b) *along any path* but not equal to (a,b).

7.2 Investigation of Pathways

Let's look at the following example to highlight this definition:

Example 7.2.1 Investigate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

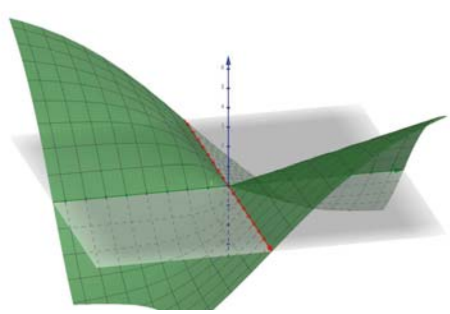

Figure 7.2.1: Graph of $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

View in Geogebra: https://www.geogebra.org/3d/wmwdgdmx

Exercise 7.2.1 Investigate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.

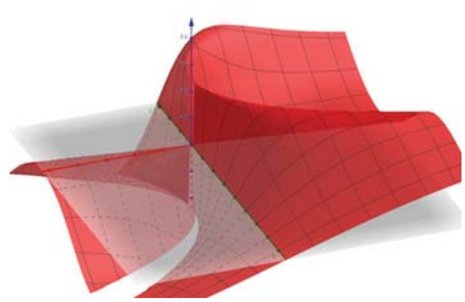

Figure 7.2.2: Graph of $f(x,y) = \dfrac{xy^2}{x^2+y^4}$
View in Geogebra: https://www.geogebra.org/3d/cmp7y8wm

7.3 Proving a Limit

Example 7.3.1 In example 7.2.1 we investigated $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$ and from that investigation gained evidence that this limit equals 0. Here we prove this fact.

Exercise 7.3.1 Prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{2x^2 + y^2} = 0$$

7.4 Continuity of Functions in Two Variables

As with our introduction to limits in two variables, let us compare continuity of functions in two variables to the continuity of a function in a single variable.

Definition 7.4.1

A function, f, of a single variable is said to be **continuous at** $x = a$ if and only if

$$\lim_{x \to a} f(x) = f(a).$$

We say f is continuous on some interval, I, if and only if f is continuous at every x in I.

That is, we say a function is continuous at an x-value when the limit equals the output of the function and say a function is continuous on some interval when it is continuous at every x-value in that interval.

When we are looking at a function in two variables we want a definition which is analogous to the single variable definition. We want the surface of our two variable function to be smooth without having to "pick up" our pencil. We don't want any instantaneous jumps at any point from any direction. Thus:

Definition 7.4.2

A function, f, in two variables is said to be **continuous at** $(x, y) = (a, b)$ if and only if

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b).$$

We say f is continuous on some domain, $D \subseteq \mathbb{R}^2$, if and only if f is continuous at every point (a, b) in D.

For our purposes, we want to be able to quickly look at a function in two-variables and determine the set of points for where it is continuous. To do this let's note that the following basic functions are continuous *on their domain*:

- Polynomial Functions
- Radical Functions
- Rational Functions
- Exponential Functions
- Logarithmic Functions
- Trigonometric Functions

With this in mind, we can use the following theorem to determine the set of points where many functions are continuous.

Theorem 7.4.1

Suppose functions f and g are both continuous on their domain. Then the following functions are also continuous on their domain:

- $f \cdot g$
- $af + bg$ where a and b are any real numbers.
- f/g
- $f \circ g$

This theorem is a little bit redundant as we could use the list of basic continuous functions and that a composition $(f \circ g)$ of continuous functions is also continuous to derive the other three facts, but it is nice to state them explicitly.

NOTE: The important thing to remember about continuity of functions is that you have to look at the domain of the composite function which may be different than the domains of either of the original functions!

Example 7.4.1 Let $H(x, y) = \dfrac{e^x + e^y}{e^{xy} - 1}$

a. Determine the set of points for which the function H is continuous.

b. Find $\displaystyle\lim_{(x,y)\to(1,1)} H(x, y)$.

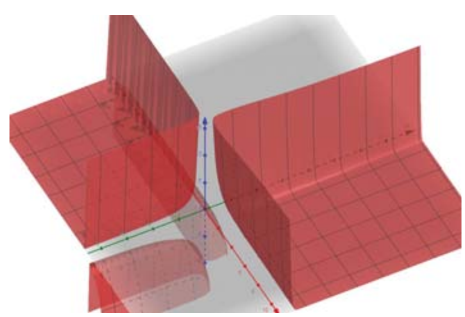

Figure 7.4.1: Graph of $H(x, y) = \dfrac{e^x + e^y}{e^{xy} - 1}$

View in Geogebra: https://www.geogebra.org/3d/gz6qn9ds

Exercise 7.4.1 Let $G(x,y) = \sin\left(\dfrac{x^2+y}{x+y}\right)$

a. Determine the set of points for which the function G is continuous.

b. Find $\displaystyle\lim_{(x,y)\to(\pi,0)} G(x,y)$.

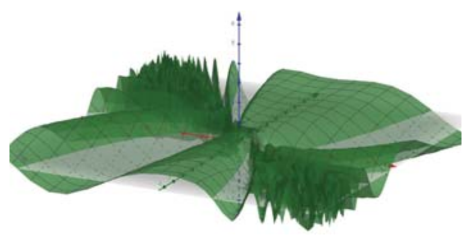

Figure 7.4.2: Graph of $G(x,y) = \sin\left(\dfrac{x^2+y}{x+y}\right)$
View in Geogebra: **https://www.geogebra.org/3d/zqez5ckd**

7.5 Continuity of Piecewise Functions in Two Variables

The great thing about Theorem 7.4.1 is that it takes care of "most" functions that can be written in an explicit algebraic form. However, when we enter the world of piecewise functions, we are no longer in a situation that Theorem 7.4.1 can handle becuase of the freedom of possibility that piecewise functions allow. When creating a piecewise function one may create all sorts of discontinuities by playing with the input expressions for different sets of points in $\mathbb{R}^2$. Here we take a look at one such example.

Example 7.5.1 Determine the set of points for which

$$f(x,y) = \begin{cases} \dfrac{xy^2}{2x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous.