LESSON 5

Parametric Surfaces

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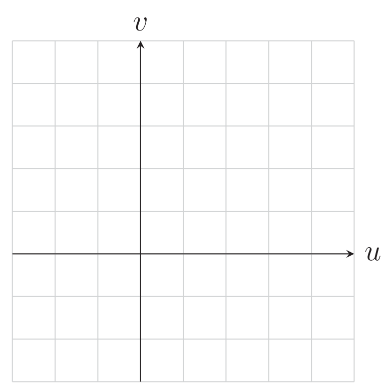
5.1 Introduction and Definitions

We have been investigating Vector-Valued-Functions of a single variable which produce Space Curves, or one-dimensional objects (lines) in three-dimensional space. We now look at Vector-Valued-Functions of two-variables and note that these will produce two-dimensional objects (surfaces) in three-dimensional space.

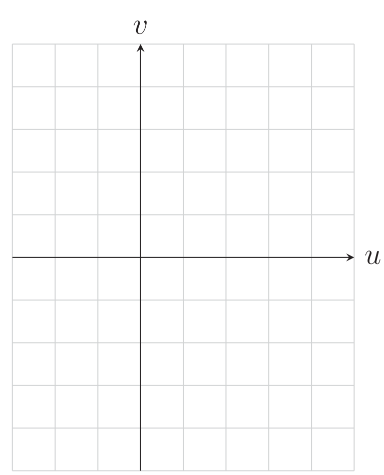
Definition 5.1.1

A **parametric surface** in three-dimensions is the set of all points (x, y, z) in $\mathbb{R}$ satisfying $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ for two real parameters u and v in some domain, D. This surface may be described by a vector function, $\mathbf{r}(u, v)$, in two variables.

Example 5.1.1 Given $\mathbf{r}(u, v) = \ln(u)\mathbf{i} + \sqrt{v}\mathbf{j} + u \cdot v\mathbf{k}$, determine and sketch the domain of $\mathbf{r}$.



Exercise 5.1.1 Given $\mathbf{r}(u, v) = \dfrac{1}{uv}\mathbf{i} - \sin(u)\mathbf{j} + \sqrt{u} \cdot \mathbf{k}$, determine and sketch the domain of $\mathbf{r}$.



Example 5.1.2 Given $\mathbf{r}(u, v) = \langle u + v, u^2 - v, u + v^2 \rangle$, determine if the point $P = (7, 5, 19)$ is on its surface.

Exercise 5.1.2 Given $\mathbf{r}(u, v) = \langle u^2 + v, u - v, v^2 - u \rangle$, determine if $P = (7, 5, 0)$ is on its surface.

5.2 Identifying and Plotting Parametric Surfaces

We will investigate a few different ways of identifying and plotting parametric surfaces, and to start we want to compare them to known equations of surfaces in three variables. Let us review some known equations of surfaces so that we can see how the parametric equations can be translated into known equations.

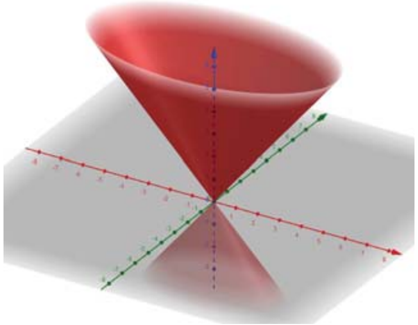


Figure 5.2.1: Cone
$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
View Graph Using Geogebra
https://www.geogebra.org/3d/pkpjxemv

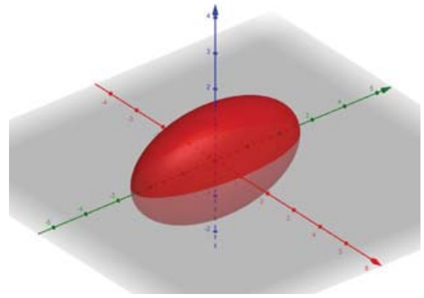


Figure 5.2.2: Ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
View Graph Using Geogebra
https://www.geogebra.org/3d/eku5kxa7

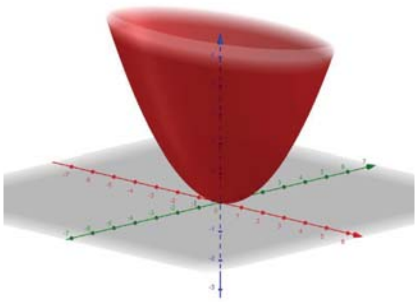


Figure 5.2.3: Elliptic Paraboloid
$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
View Graph Using Geogebra
https://www.geogebra.org/3d/jmkbfgnj

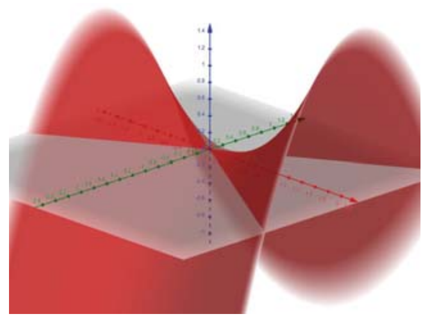


Figure 5.2.4: Hyperbolic Paraboloid
$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
View Graph Using Geogebra
https://www.geogebra.org/3d/zdebsjvk

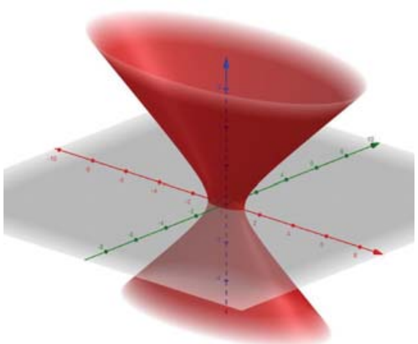


Figure 5.2.5: Hyperboloid of One Sheet
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
View Graph Using Geogebra
https://www.geogebra.org/3d/zuqcxpdf

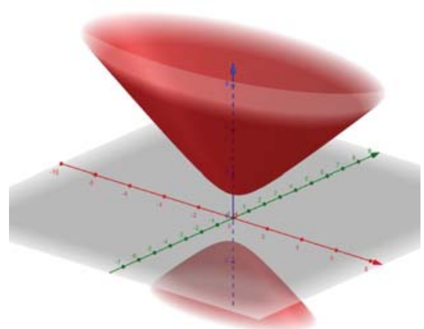
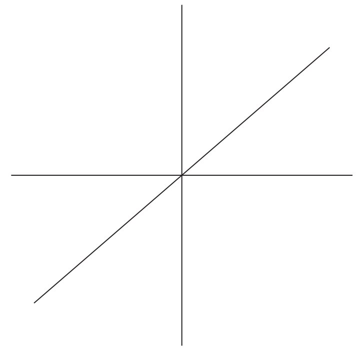
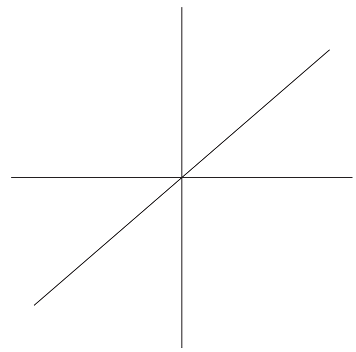


Figure 5.2.6: Hyperboloid of Two Sheets
$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
View Graph Using Geogebra
https://www.geogebra.org/3d/vepzha2d

Example 5.2.1 Identify and sketch the surface given by $\mathbf{r}(u, v) = \langle u, v, u^2 - v^2 \rangle$.



Exercise 5.2.1 Identify and sketch the surface given by $\mathbf{r}(u, v) = \langle \cos(u), v, \sin(u) \rangle$.



Definition 5.2.1

The **grid curves** of a parametric surface are the lines created when one of the parameters of the Vector-Valued-Function-in-Two-Parameters is held constant so that it becomes a Vector-Valued-Function-in-One-Parameter.

Note that we can use Geogebra to graph parametric surfaces. While I will use grid curves in the next example to sketch the surface so you can see how we can do this without a computer, I've also graphed the surface in Geogebra so that we can see the grid curves there as well and confirm the accuracy of my sketch. The syntax for entering this parametric surface into Geogebra is

$$\begin{gathered}\mathrm{r}(u, v) = \text{surface}(x - \text{component}, y - \text{component}, z - \text{component}, \\ \text{1st Input Paremter}, \text{1st Input Paremeter Start-Value}, \text{1st Input Paremeter End-Value}, \\ \text{2nd Input Paremeter}, \text{2nd Input Paremeter Start-Value}, \text{2nd Input Paremeter End-Value})\end{gathered}$$

and, for the following example, this will look like

$$r(u, v) = \text{surface}(u^2, v - u, v^2 - u, u, -10, 10, v, -10, 10).$$

Example 5.2.2 Let $\mathbf{r}(u, v) = \langle 2u, u^2 + v^2, 3v \rangle$. Use grid curves to sketch the parametric surface.

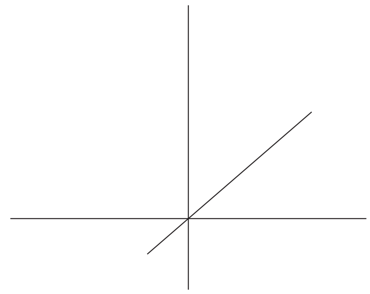


Figure 5.2.7: View Graph Using Geogebra
https://www.geogebra.org/3d/efzrb4z2

Exercise 5.2.2 Match the following vector equations to their respective graphs.

(a) $\mathbf{p}(u, v) = \langle u, v, u \cdot v \rangle$

(b) $\mathbf{q}(u, v) = \langle u^2 \cdot \cos(v), u^2 \cdot \sin(v), u \rangle$

(c) $\mathbf{r}(u, v) = \langle 2\sin(v)\cos(u), 2\sin(v)\sin(u), 2\cos(v) \rangle$

(d) $\mathbf{s}(u, v) = \langle \sin(v), \cos(u)\sin(2v), \sin(u)\sin(2v) \rangle$ on $-\pi \leq v \leq \pi$

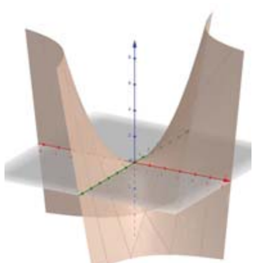


Figure 5.2.8: I
View Graph Using Geogebra
https://www.geogebra.org/3d/nm6cvpxm

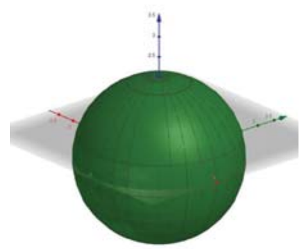


Figure 5.2.9: II
View Graph Using Geogebra
https://www.geogebra.org/3d/kcbwyqg7

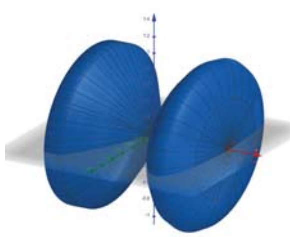


Figure 5.2.10: III
View Graph Using Geogebra
https://www.geogebra.org/3d/b5va3kpv

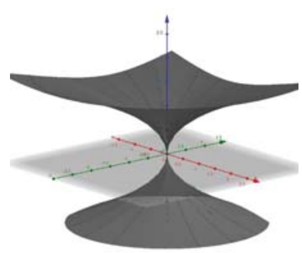


Figure 5.2.11: IV
View Graph Using Geogebra
https://www.geogebra.org/3d/pdpjrwhk

5.3 Parametrizing Equations of Surfaces

It is sometimes convenient to take the equation of a surface and to turn it into a vector function in two variables to be able to analyze in a different manner or to satisfy the requirements of some software you are using. Or, you may have the idea for a shape without an equation and you want to be able to come up with an vector function in two variables to represent it. This can also be useful for determining volumes of objects which we will look at in later chapters.

Example 5.3.1 Find a parametric representation for the part of the elliptic paraboloid $x + y^2 + 2z^2 = 4$ that lies in front of the plane $x = 0$.

Example 5.3.2 Parameterize a circular cone with an interior angle of 45°.

Exercise 5.3.1 Find parametric equations for the surface obtained by rotating the curve $y = e^{-x}$, over the interval $0 \leq x \leq 3$, about the x-axis and use them to graph the surface.

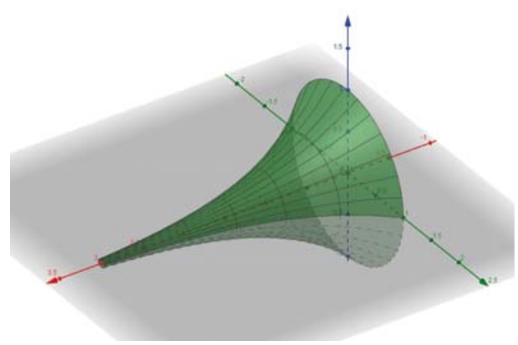


Figure 5.3.1: View Graph Using Geogebra
https://www.geogebra.org/3d/ad5uad9x