LESSON 3

Arc Length and Curvature

Contents

In response to plotting space curves, we raise the question of how long a curve is over some interval. In the case of a space curve following the path of a moving object, the question could be posed as how far the object travels over a given interval of time. It is also of interest to discuss the curvature of the space curve (how tightly it turns) at different points along its path and to have a couple of different quantitative ways of measuring this.

3.1 Arc Length

To determine a formula to give us the arc length of a space curve over some interval we will build a Riemann Sum out of the lengths of higher and higher quality straight-line approximations to the curve.

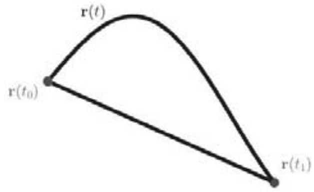

Figure 3.1.1: Arc Length Approximation using two points.

We begin by taking a distance approximation using just the endpoints, which gives

$$D = |\mathbf{r}(t_1) - \mathbf{r}(t_0)|.$$

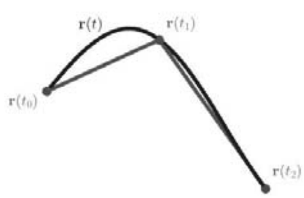

Figure 3.1.2: Arc Length Approximation using three points.

We then re-approximate the distance using three points using

$$\begin{aligned} D &= |\mathbf{r}(t_1) - \mathbf{r}(t_0)| + |\mathbf{r}(t_2) - \mathbf{r}(t_1)| \\ &= \sum_{i=1}^{2} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|. \end{aligned}$$

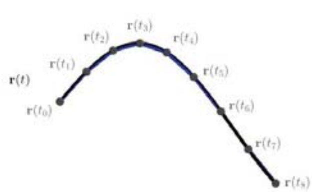

Figure 3.1.3: Arc Length Approximation using eight points.

Repeating this using eight points then yields

$$D = \sum_{i=1}^{8} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|.$$

Note then that if n equals the number of straight lines the curve is approximated by, we find that:

$$\Delta t = \frac{t_n - t_0}{n}, \quad t_i = t_0 + \Delta t \cdot i, \quad i \in \{1, 2, ..., n\}.$$

Letting $\Delta \mathbf{r}_i = \mathbf{r}(t_i) - \mathbf{r}(t_{i-1})$ we get the distance approximation using n straight lines and equal t-spacing of:

$$D = \sum_{i=1}^{n} |\Delta \mathbf{r}_i| = \sum_{i=1}^{n} \left| \frac{\Delta \mathbf{r}_i}{\Delta t} \right| \Delta t.$$

We want to take the limit as $n \to \infty$ to get an exact value for the distance along the curve which then results in our final formula:

$$\begin{aligned} D &= \lim_{n\to\infty} \sum_{i=1}^{n} \left| \frac{\Delta \mathbf{r}_i}{\Delta t} \right| \Delta t \\ &= \int_{t_0}^{t_f} |\mathbf{r}'(t)| dt \\ &= \int_{t_0}^{t_f} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \end{aligned}$$

where t_f is the t-value of the right end-point of the curve.

Example 3.1.1 Find the length of the curve $\mathbf{r}(t) = \langle 2\sin(t), 2\cos(t), 5t \rangle$ over $0 \le t \le \pi$.

Exercise 3.1.1 Find the length of the curve $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, \frac{1}{9}(6t)^{3/2}, 3t \right\rangle$, over $1 \le t \le 4$.

3.1.1 Reparameterization of a Vector Function from t to arc length, s

Note that when we define a space-curve with an input of t, we will almost always see that a single unit of t (which we often think of as time) will *not* correspond to a single unit length. However, it is often possible to redefine the curve in such a way where the input is the length of the curve traveled from the starting point (which we often symbolize with s).

Example 3.1.2 Reparameterize the curve $\mathbf{r}(t) = 2t\mathbf{i} + (1 - 3t)\mathbf{j} + (5 + 4t)\mathbf{k}$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

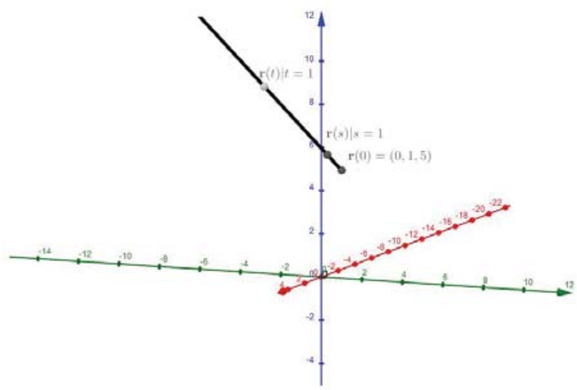

Figure 3.1.4: View Graph Using Geogebra
https://www.geogebra.org/3d/j9mbbwyn

Exercise 3.1.2 Reparameterize the curve $\mathbf{r}(t) = \langle \sin(t), 3t, \cos(t) \rangle$ with respect to arc length measured from the point where $t = 0$ in direction of increasing t.

3.2 Curvature

If we consider a measurement for the amount of curvature on a curve, our intuition would conclude that a straight line will have curvature 0 because the tangent vector is constant, while small circles will have large curvature and large circles will have small curvature. If we consider unit-tangent vectors along a curve, if there is a sharp turn, the unit-tangent vector changes more quickly while if the turn is 'smoother,' the tangent vector changes more slowly.

We may now define the curvature κ (lower-case, Greek-letter kappa) of a space curve C.

Definition 3.2.1

The **curvature** κ (lower-case, Greek-letter kappa) of a space curve C with $\mathbf{T}$ as the unit tangent vector and $s(t)$ as the arc-length function is $\kappa = \left|\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|$

There are variations to curvature formulas for κ:

$$\begin{aligned}\kappa &= \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| \\ &= \left|\frac{\mathbf{T}'(t)}{s'(t)}\right| \\ &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}\end{aligned}$$

The last version of κ is more commonly used for calculations. The proof is left as a reading from the text.

We can interpret the curvature κ as the magnitude of the rate of change of the unit tangent vector with respect to the arc length.

Example 3.2.1 Find the curvature of $\mathbf{r}(t) = \langle t^2, \ln(t), t\ln(t)\rangle$ at the point $(1, 0, 0)$. Note: this will give the measure of how quickly the curve changes direction at the given point.

Exercise 3.2.1 Find the curvature of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$

3.3 Unit Tangent and Unit Normal Vectors

It is sometimes useful to look at a given point on a curve, to be able to determine the tangent vector at that point, and then be able to adjust the tangent vector so that it has length one. If we can do this generally for any point we acquire what is called the **Unit Tangent Function**, denoted $\mathbf{T}(t)$, which gives the direction of the curve at any point in time.
Another useful function is called the **Unit Normal Function**, denoted $\mathbf{N}(t)$, which gives a unit vector pointing at a right-angle to the tangent vector and to the inside of the curve. That is, it points in the direction that the curve *is turning*. Thus, by coming up with this function we can quickly tell the direction the curve is turning at any point in time.

Definition 3.3.1

The **Unit Tangent Function** for a Vector Function $\mathbf{r}$, is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|},$$

which gives the direction of $\mathbf{r}(t)$ at any point along the curve.

Note that this is a somewhat intuitive calculation since $\mathbf{r}'(t)$ is the direction-vector of the curve at any point and by dividing by $|\mathbf{r}'(t)|$ we change the magnitude of this vector to be 1.

Definition 3.3.2

The **Unit Normal Function** for a Vector Function $\mathbf{r}$, is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|},$$

which gives the direction that $\mathbf{r}(t)$ is turning at any point along the curve.

Here this is again somewhat intuitive as $\mathbf{T}'(t)$ is the rate of change of the direction-vector, hence producing a vector in the direction of the change in direction. Again, by dividing by $|\mathbf{T}'(t)|$ we are simply changing the magnitude of this change-in-direction vector to be 1.

Example 3.3.1 Find the unit tangent and unit normal vectors, $\mathbf{T}(t)$ and $\mathbf{N}(t)$ respectively, for the Vector Function $\mathbf{r}(t) = \langle t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t)\rangle$ at the point $(\pi^2, \pi, -1)$ on the curve.

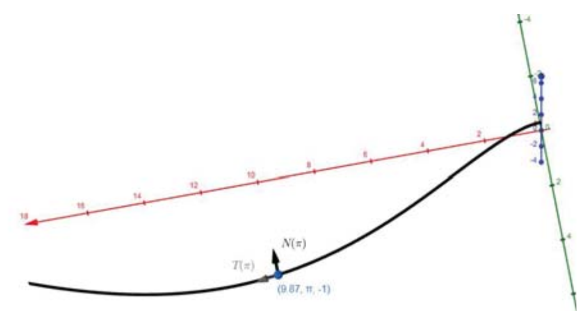

Figure 3.3.1: View Graph Using Geogebra
https://www.geogebra.org/3d/ds7gwjpt

Exercise 3.3.1 Find the Unit Tangent and Unit Normal Vector Functions $\mathbf{T}(t)$ and $\mathbf{N}(t)$ respectively, for the Vector Function $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, t^2 \right\rangle$

3.4 Equations of Normal Planes to a Curve

Recall the equation of a plane is given by

$$ax + by + cz = d,$$

where $\langle a, b, c\rangle$ is any normal vector to the plane (a vector orthogonal to the plane) and d can then be found by substituting any point into the equation for the plane and simplifying.

Definition 3.4.1

The **Normal Plane** to a curve at a given point is the plane which contains the point in question and whose normal vector is the tangent vector to the curve at that point.

Example 3.4.1 Find the equation of the normal plane to the curve $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ at the point $(1, 1, 1)$.

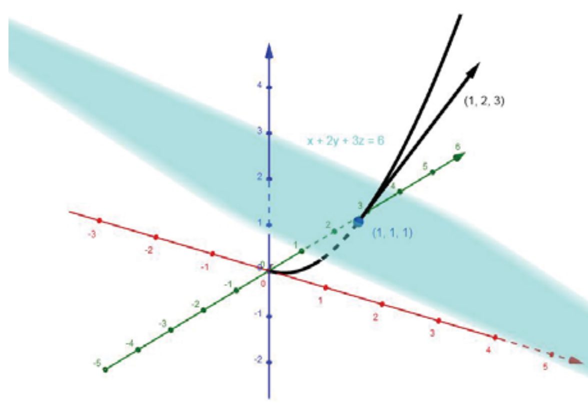

Figure 3.4.1: View Graph Using Geogebra
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Exercise 3.4.1 Find the equation of the normal plane to the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ at the point $P = (0, 1, \pi/2)$.

3.5 Length of Intersection of Two Surfaces

As a final extension to the ideas introduced in this lesson, we review how to find the curve of intersection of two surfaces and, supposing this curve is of finite length, to then determine that length.

Example 3.5.1 Find the length of the curve of intersection of the cylinder $4x^2 + y^2 = 4$ and the plane $x + y + z = 2$.

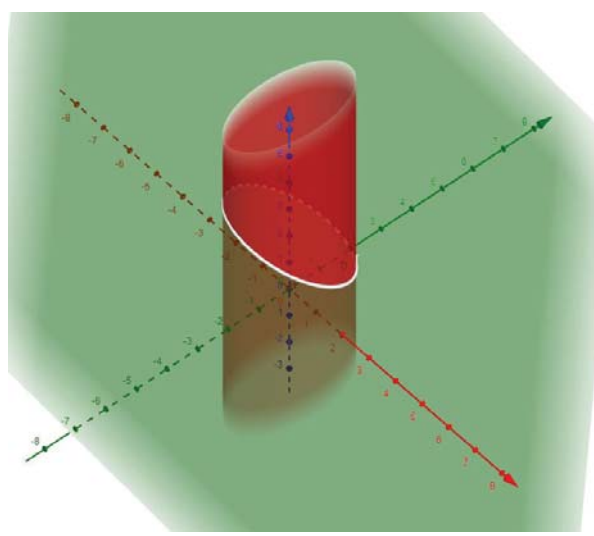

Figure 3.5.1: View Graph Using Geogebra
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