LESSON 2

Derivatives and Integrals of Vector Functions

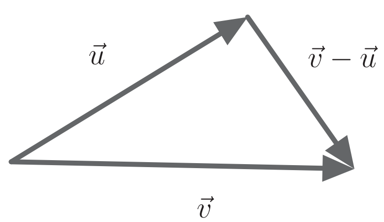
Contents

In Differential Calculus and Integral Calculus, we explored the ideas of differentiation and integration for functions of a single variable that output real numbers. We now extend these concepts to that of vector-valued functions. While the derivative of a vector-valued function has a great correspondence to the derivative of a real-valued function - that is, it gives the direction of the tangent line - the integral of of a vector-valued function is a little bit more abstract as it will produce another vector-valued function, not an area under a curve per se. We will explore some of the applications of this in later sections.

2.1 Tangent Vectors

2.1.1 Background Review

Recall the basic vector operation of subtraction.



Now let $\mathbf{r}(t)$ be a vector function.

In the analogous step from differentiation in one-variable, we define $\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$.

The procedure for differentiating $\mathbf{r}(t)$ occurs in the natural way — by component.

2.1.2 Derivatives

Example 2.1.1 Find the derivative of the vector function $\mathbf{r}(t) = \langle \cos(3t), t, \sin(3t) \rangle$

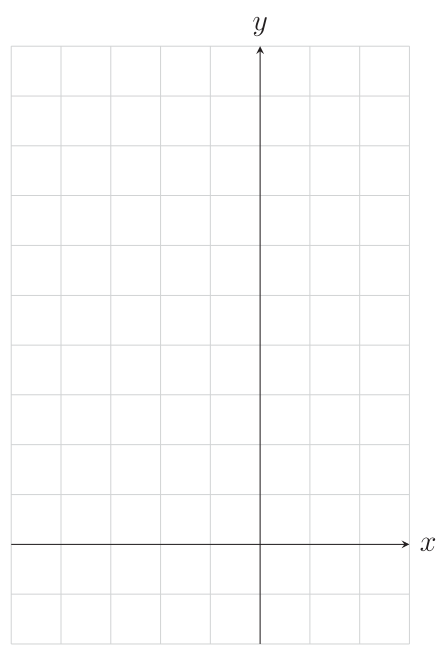
Definition 2.1.1

The **Tangent Vector** to a vector-valued function, $\mathbf{r}(t)$ at a given point P, with P on $\mathbf{r}$ at $t = a$, is the vector $\mathbf{r}'(a)$ with initial point at $\mathbf{r}(a)$. The tangent vector at P exists if $\mathbf{r}'(a)$ exists and $\mathbf{r}'(a) \neq \mathbf{0}$.

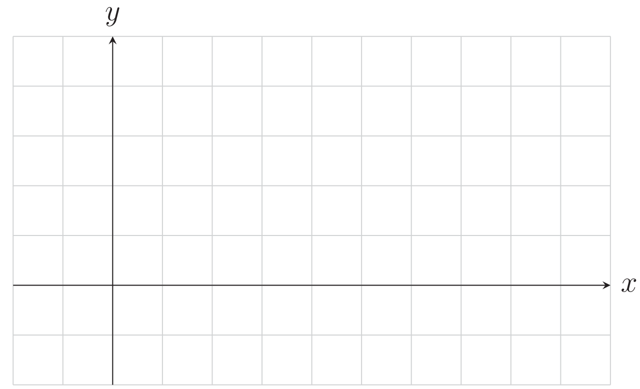
Example 2.1.2 Find the tangent vector to $\mathbf{r}(t) = \langle \cos(3t), t, \sin(3t) \rangle$ at the point when $t = \pi$

2.2 Tangent Lines

Example 2.2.1 Given the vector-valued function $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle$, sketch the plane-curve, find $\mathbf{r}(t)$, sketch the position vectors for $\mathbf{r}(-2)$, $\mathbf{r}(-1)$, and $\mathbf{r}(0)$, sketch the tangent vector $\mathbf{r}'(-1)$, and determine the equation of the tangent line at this location.



Exercise 2.2.1 Given the vector-valued function $\mathbf{r}(t) = \langle t+1, \sqrt{t}\rangle$, sketch the plane-curve, find $\mathbf{r}(t)$, sketch the position vectors for $\mathbf{r}(1)$, $\mathbf{r}(2)$, and $\mathbf{r}(3)$, sketch the tangent vector $\mathbf{r}'(2)$, and determine the equation of the tangent line at this location.



Example 2.2.2 Given the vector-valued function $\mathbf{r}(t) = \langle 4\sqrt{t}, t, t^2\rangle$ find $\mathbf{r}'(t)$ then determine the equation of the tangent line at the point $(4, 1, 1)$. Use technology to graph the space curve and tangent line at this point.

To plot these using Geogebra, we will enter

(4sqrt(t),t,t∧2)

to produce the space curve. We then plot the point that we want to produce a tangent line from by entering $(4, 1, 1)$ as the second input. We can enter

$$\text{Vector}((4,1,1),(6,2,3))$$

to create the tangent vector from the given point. Finally, we enter

$$(4,1,1) + (2,1,2)t$$

to plot the entire tangent line.

Confirm Graph Using Geogebra https://www.geogebra.org/3d/kgrgzvsa

Exercise 2.2.2 Given the vector-valued function $\mathbf{r}(t) = \langle t\sin(t), t^2, t\cos(2t)\rangle$ find $\mathbf{r}'(t)$ then determine the equation of the tangent line at the point $(0, 0, 0)$. Use technology to graph the space curve and tangent line at this point.

Confirm Graph Using Geogebra https://www.geogebra.org/3d/njgcywym

2.3 Unit Tangent Vectors

Definition 2.3.1

The **Unit Tangent Vector** is a vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ that has length 1.

Example 2.3.1 Find the unit tangent vector $\mathbf{T}(t)$ to the vector-valued function

$$\mathbf{r}(t) = 2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j} + \tan(t)\mathbf{k}$$

at $t = \frac{\pi}{4}$.

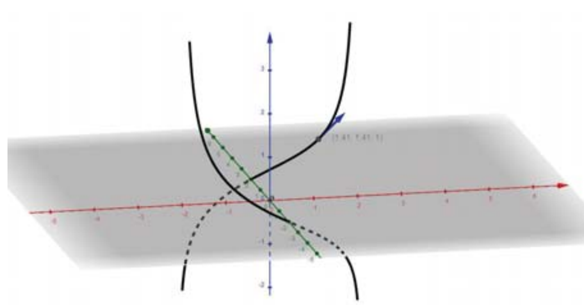


Figure 2.3.1: View Graph Using Geogebra https://www.geogebra.org/3d/mekypqct

Exercise 2.3.1 Find the unit tangent vector $\mathbf{T}(t)$ to the vector-valued function

$$\mathbf{r}(t) = \sin(2t)\mathbf{i} + 2t\mathbf{j} + \cos(t)\mathbf{k}$$

at $t = \frac{\pi}{2}$.

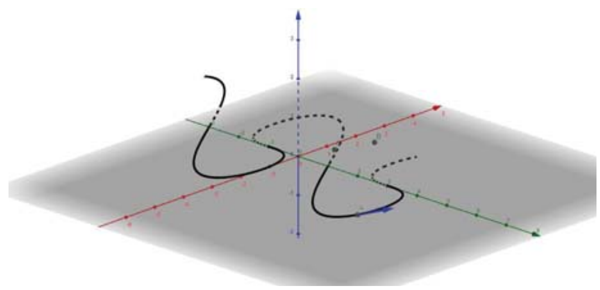


Figure 2.3.2: View Graph Using Geogebra
https://www.geogebra.org/3d/ma5tkcaj

Example 2.3.2 Find the tangent vector and unit tangent vector to the curve with parametric equations

$$x = \cos(t), y = \sin(t), z = t$$

at the point (1,0,0) then determine the parametric equations for the tangent line at this point. Use technology to plot the space curve, tangent vector, unit tangent vector, and tangent line.

Geogebra instructions on next page.

To use Geogebra to plot the given objects, we start by inputting

(cos(t),sin(t),t)

in the first input box and change the starting t-value to -2π.
To generate the tangent vector, place the point by inputting into the second box $(1, 0, 0)$ and then input $a'(0)$, since $t = 0$ is where the point $(1, 0, 0)$ occurs, to get the tangent vector direction. To plot this as the tangent vector we enter

Vector(A,A+B)

so that Geogebra plots the vector from the point A in the direction $(0, 1, 1)$.
To plot the unit tangent vector, we then enter $\mathrm{C} = \mathrm{B}/|\mathrm{B}|$ to generate the unit tangent vector and, then in the next input box, enter

Vector(A,A+C)

. Finally, to plot the tangent line, we'll enter

$$\mathrm{X} = \mathrm{A} + t * \mathrm{C}$$

.

Confirm Graph Using Geogebra https://www.geogebra.org/3d/gd47kxjq

2.4 Angles of Intersection

Example 2.4.1 At what point do the curves $\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$ and $\mathbf{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$ intersect? Find their angle of intersection.

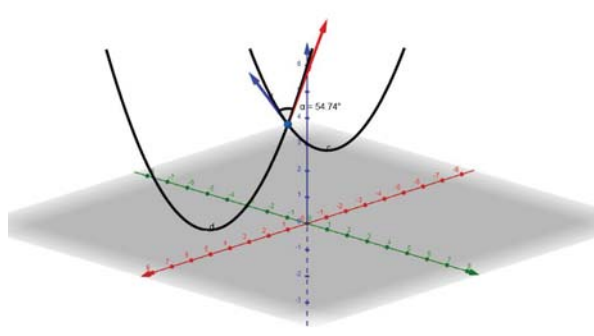


Figure 2.4.1: View Graph Using Geogebra
https://www.geogebra.org/3d/xbr3kubs

2.5 Derivative formulas

In general, the derivative formulas for real-valued functions are similar for vector-valued functions. Below are a few commonly used formulas for differentiation.

Theorem 2.5.1

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and c is a scalar, then

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}(c\mathbf{u}(t)) = c\mathbf{u}'(t)$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

These can be proven using the component form for each vector.

Example 2.5.1 Find $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) \times \mathbf{v}(t))$ for $\mathbf{u}(t) = 2t\mathbf{i} + 6t\mathbf{j} + t^2\mathbf{k}$ and $\mathbf{v}(t) = e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}$.

Exercise 2.5.1 For vectors $\mathbf{u}$ and $\mathbf{v}$ in Example 2.5.1, find $\mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$. How do the results compare with Example 2.5.1 and Theorem 2.5.1(d)?

2.6 Integration of Vector Valued Functions

Definite integrals of vector-valued functions may now be considered. Specifically, if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f, g, and h are integrable on $[a, b]$, then by definition,

$$\begin{aligned}\int_a^b \mathbf{r}(t)dt &= \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k} \\ &= \mathbf{R}(t)\big|_a^b \\ &= \mathbf{R}(b) - \mathbf{R}(a)\end{aligned}$$

from the Fundamental Theorem of Calculus such that $\mathbf{R}'(t) = \mathbf{r}(t)$.

Likewise, if $\mathbf{R}'(t) = \mathbf{r}(t)$, then every antiderivative of $\mathbf{r}(t)$ is of the form $\mathbf{R}(t) + \mathbf{C}$ for some constant vector $\mathbf{C}$. We can therefore write $\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{C}$ if $\mathbf{R}'(t) = \mathbf{r}(t)$

Example 2.6.1 Evaluate $\displaystyle\int_0^1 \left(\frac{4}{1+t^2}\mathbf{j} + \frac{2t}{1+t^2}\mathbf{k}\right) dt$.

Exercise 2.6.1 Evaluate $\displaystyle\int (\sec^2(t)\mathbf{i} + t(t^2+1)^3\mathbf{j} + t^2\ln(t)\mathbf{k})dt$.