TAYLOR'S INEQUALITY

MTH 253 LECTURE NOTES

Note: We can find a Taylor series for several functions, but how do we know that this Taylor series *actually equals* the function? We made an assumption about f being able to be represented by a power series, so now let's explore *when* this assumption is valid.

Definition

If we let $R_n(x) = f(x) - T_n(x)$, then we call $R_n(x)$ the **remainder** of the Taylor polynomial $T_n(x)$.

Example 1. Find the remainder for the third-degree Taylor polynomial for $f(x) = e^x$ centered at 0.

Theorem

If $f(x) = T_n(x) + R_n(x)$ and $\lim_{n\to\infty} R_n(x) = 0$ for $|x - a| < R$, then $f(x) = \lim_{n\to\infty} T_n(x)$ on the interval $|x - a| < R$.

Proof:

Theorem

Weighted Mean Value Theorem for Integrals
If f and g are continuous on $[a, b]$ and g does not change sign on $[a, b]$, then there exists a number $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\;dx = f(c)\int_a^b g(x)\;dx$$

Proof:

Theorem

If $f^{(n+1)}$ is continuous on an open interval I containing a, and $x \in I$, then there exists some value $c \in (a, x)$ such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

and we call this the **Lagrange form of the Remainder Term**.

Proof:

Theorem

Taylor's Inequality
If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor polynomial satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for $|x-a| \leq d$.

Proof:

Example 2. Prove that e^x is equal to the sum of its Maclaurin series.

Exercise 1. Show that $f(x) = \sin x$ is equal to the sum of its Maclaurin series.

Example 3. Show that $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$.