TAYLOR & MACLAURIN SERIES

MTH 253 LECTURE NOTES

Exploration:

If f can be represented by a power series centered at a, then

$$f(x) =$$

$$f'(x) =$$

$$f''(x) =$$

$$f'''(x) =$$

$$\vdots$$

Evaluate each of the previous functions at a:

$$f(a) =$$

$$f'(a) =$$

$$f''(a) =$$

$$f'''(a) =$$

$$\vdots$$

$$f^{(n)}(a) =$$

And so we can conclude

$$c_n =$$

Theorem

If we can write $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ with $|x-a| < R$, then $c_n = \dfrac{f^{(n)}(a)}{n!}$. Thus,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

Definition

The series representation of $f(x)$ given by $\sum_{n=0}^{\infty} c_n (x-a)^n$ is called the **Taylor series for f at a**.

Definition

The Taylor series for f at 0 is called the **Maclaurin series for f**. Thus,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \end{aligned}$$

is the Maclaurin series for f.

Example 1.
Find the Maclaurin series for $f(x) = e^x$ and its radius of convergence.

Note : Since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, it must be the case that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

Definition

The polynomial

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the n**th-degree Taylor polynomial for f at a.**

Note : A Taylor polynomial of degree n is a partial sum of the Taylor series.

Example 2. Find the first, second, and third-degree Taylor polynomials for $f(x) = e^x$ at 0.

Exercise 1. Let $f(x) = \sin x$

(a) Find a Maclaurin series for $f(x)$.

(b) Find the radius of convergence of the Maclaurin series you found in part.

(c) Find the 5th degree Taylor polynomial for $f(x)$ at 0.

Example 3. Find the Taylor series for $f(x) = e^x$ at $a = 10$.

Table of Known Maclaurin Series

Function	Maclaurin Series		Radius of Convergence
$\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n$	$= 1 + x + x^2 + x^3 + \cdots$	$R = 1$
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$= 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$= x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$= 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$R = \infty$

Example 4. Evaluate $\displaystyle\int e^{-x^2}\, dx$ as an infinite series.

Example 5. Find the Maclaurin series for $f(x) = (1+x)^p$, where $p \in \mathbb{R}$. Then find its interval of convergence.

Definition

A **binomial coefficient** is defined as

$$\binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}$$

and is read as "p choose n" due to its applications in probability.

Definition

If $p \in \mathbb{R}$ and $|x| < 1$, then the **Binomial Series** for $(1+x)^p$ is

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

Exercise 2. Find the Maclaurin series for $f(x) = \sqrt{1+x}$ and find its interval of convergence.

Example 6. Find the Maclaurin series for $f(x) = \sqrt[3]{2+x}$ and find its interval of convergence.