POWER SERIES

MTH 253 LECTURE NOTES

Definition

A **Power Series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and c_n are the coefficients of the series. The sum of a power series is a function whose domain is

$$\left\{ x \in \mathbb{R} \mid f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ converges} \right\}$$

Example 1. What is the domain of the power series given below?

$$g(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Definition

The series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a **Power Series Centered at** a (or a **Power Series About** a).

Note: We adopt the convention that $(x-a)^0 = 1$, even when $x = a$.

Example 2. What is the domain of the function $z(x) = \sum_{n=0}^{\infty} n! x^n$?

Example 3. What is the domain of $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$?

Example 4. What is the domain of $f(x) = \sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$? What is the center of this power series?

Exercise 1. What is the domain of $h(x) = \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$?

Technology Exploration: Graph the first ten partial sums of $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ in Desmos. What familiar function does $C(x)$ look like? Does the domain found in the previous example seem accurate graphically?

Theorem

Given a power series centered at a, $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

i. The series converges only when $x = a$.

ii. The series converges for all $x \in \mathbb{R}$.

iii. There exists a positive number R such that
- The series converges for all $|x-a| < R$,
- The series diverges for all $|x-a| > R$, and
- The series may converge or may diverge when $|x-a| = R$.

Definition

The number R in the theorem above is called the **Radius of Convergence** of the power series. The **Interval of Convergence** is the interval for which the power series converges. In relation to the theorem above,

i. The radius of convergence is 0; the interval of convergence is $[a, a]$.

ii. The radius of convergence is ∞; the interval of convergence is $\mathbb{R} = (-\infty, \infty)$.

iii. The radius of convergence is R; the interval of convergence is either

$(a-R, a+R)$ or $(a-R, a+R]$ or $[a-R, a+R)$ or $[a-R, a+R]$

Example 5. Find the radius of convergence and interval of convergence for the following series. Then graph the domain on a number line.

a. $g(x) = \sum_{n=0}^{\infty} x^n$

b. $z(x) = \sum_{n=0}^{\infty} n!x^n$

c. $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

d. $f(x) = \sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$

Exercise 2. Find the radius of convergence and the interval of convergence for the series below. Then graph its domain on a number line.

$$h(x) = \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$