ALTERNATING SERIES

MTH 253 LECTURE NOTES

Definition

An **Alternating Series** is a series whose terms alternate between positive and negative. Alternating series, $\sum a_n$, typically have terms of the form

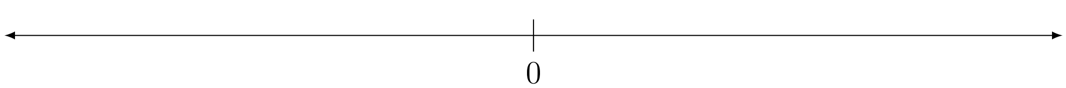
$$a_n = (-1)^n b_n \qquad \text{or} \qquad a_n = (-1)^{n-1} b_n$$

where $b_n > 0$.

Exploration: Consider an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

where the sequence $\{b_n\}$ is positive, decreasing, and whose limit is 0.



Theorem

Alternating Series Test

If $\sum_{n=1}^{\infty} a_n$ is an alternating series where $a_n = (-1)^{n-1} b_n$ or $a_n = (-1)^n b_n$, and $\{b_n\}$ is

- Positive,
- Decreasing, and
- $\lim_{n \to 0} b_n = 0$,

then the alternating series $\sum_{n=1}^{\infty} a_n$ converges.

Example 1. Determine whether the alternating harmonic series is convergent or divergent. Justify your conclusion as specifically as possible.

Example 2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is convergent or divergent. Justify your conclusion as specifically as possible.

Exercise 1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^3+1}$ is convergent or divergent. Justify your conclusion as specifically as possible.

Exploration: Estimating a series always comes out to computing s_n. From the picture on page 1, we can see that s_n is always within b_{n+1} of the true sum.

Theorem

Alternating Series Estimation Theorem

If s is the sum of an alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$, where b_n is positive, decreasing, and $\lim_{n\to\infty} b_n = 0$, then

$$|R_n| = |s - s_n| \le b_{n+1}$$

Note: Said another way, an alternating series that satisfies the aforementioned conditions can be estimated with s_n, and the approximation is always as good as the next term of the series.

Example 3. Approximate the sum of the alternating harmonic series accurate to the nearest thousandth.

Exercise 2. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges. Justify your conclusion as specifically as possible. Then determine how many terms of the series are necessary to approximate its sum so that $|R_n| < 0.0005$.