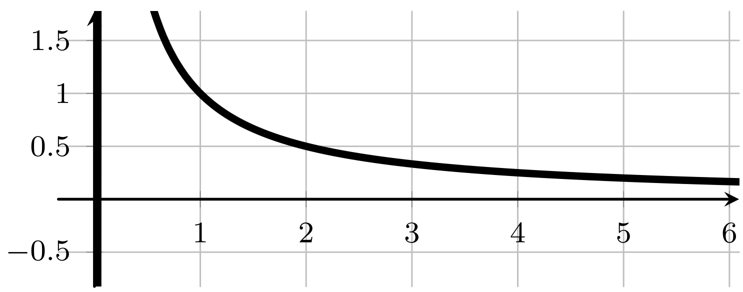
# INTEGRAL TEST

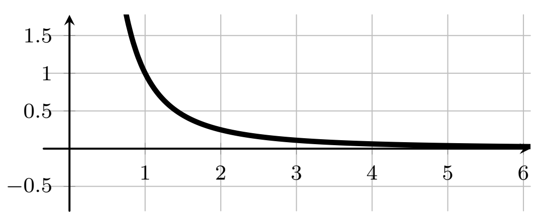
## MTH 253 LECTURE NOTES

Recall that the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Another way to prove this is to think of the series $\sum_{n=1}^{\infty} \frac{1}{n}$ as a Riemann Sum and then compare to the improper integral $\int_1^{\infty} \frac{1}{x}\, dx$.

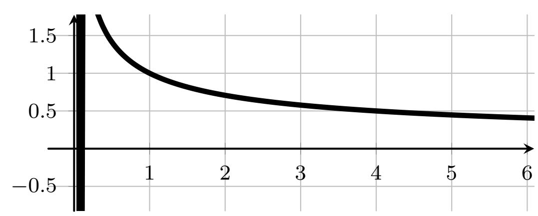



**Example 1.** Use a similar technique to determine whether each series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^2}$




b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$



**Fun Fact!:** Finding the exact value of the sum is difficult, but the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ turns out to converge to $\frac{\pi^2}{6}$. This sum was first posed in 1644 and is known as the Basel problem. It was first solved in 1734 by Leonhard Euler.

**Theorem:**

**The Integral Test**
Let $a_n = f(n)$. If $f$ is a function that is 1) continuous, 2) positive, and 3) decreasing on the interval $[1, \infty)$, then

- If the integral $\int_1^{\infty} f(x)\, dx$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.
- If the integral $\int_1^{\infty} f(x)\, dx$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

**Example 2.** Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges. Justify your conclusion as specifically as possible.

**Exercise 1.** Determine whether the series $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ converges or diverges. Justify your conclusion as specifically as possible.

**Definition:**

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a $p$-**series**.

**Exploration:** For what values of $p$ does a $p$-series definitively converge? For what values of $p$ does a $p$-series definitively diverge? Are there any values of $p$ that convergence or divergence is inconclusive?

**Theorem:**

The $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is

- Convergent if $p > 1$, and
- Divergent if $p \leq 1$.

**Note:** The Integral Test and the $p$-series test only determine convergence or divergence of a series, it does *not* find the sum of a series.

**Exercise 2.** Determine whether each series converges or diverges. Justify your conclusion as specifically as possible.

a. $\sum_{n=1}^{\infty} \frac{3}{\sqrt[3]{n}}$

c. $\sum_{n=1}^{\infty} \frac{1}{n^{-2}}$

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^7}}$

d. $\sum_{n=1}^{\infty} \frac{2}{3^n}$