ESTIMATING A SUM

MTH 253 LECTURE NOTES

Exploration: We've learned that we can always find the sum of a convergent geometric series. We can also find the sum of a convergent series if we know a rule for s_n. However, we don't know how to find the sum of a p-series or any series we determine is convergent using the Comparison Test or Limit Comparison Test. In these cases, we still may want to estimate the sum.

Recall that we can approximate *any value*, but when we do we get a corresponding error. The relationship between these three values is

$$\text{Exact} = \text{Approximation} + \text{Error}$$

In the case of a series, we will always have this relationship

$$\underbrace{\text{Sum}}_{\text{Exact}} = \underbrace{\text{Partial Sum}}_{\text{Approximation}} + \underbrace{\text{Remainder}}_{\text{Error}}$$

where s is the sum, s_n is the nth partial sum, and R_n is the remainder.

Definition

The **Remainder** of an approximation of a sum is

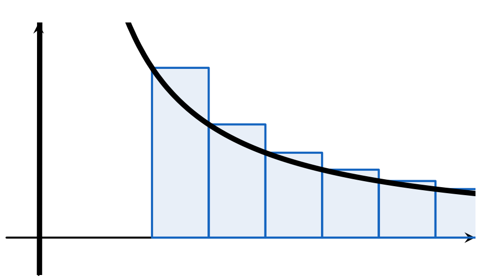
$$\begin{aligned} R_n &= s - s_n \\ &= a_{n+1} + a_{n+2} + a_{n+3} + \cdots \end{aligned}$$

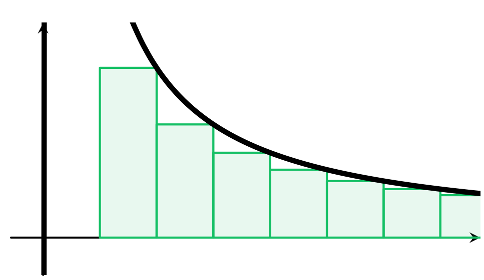
Exploration: We use approximations when an exact value is difficult or impossible to obtain. Information about the remainder can be used to tell us *how good* our approximation is.

Using the same idea as the Integral Test, we assume that f is positive, continuous, and decreasing on $[n, \infty)$. Then

$$\int_{n+1}^{\infty} f(x)\, dx \le R_n = a_{n+1} + a_{n+2} + \cdots \le \int_{n}^{\infty} f(x)\, dx$$

which is apparent from the graphs provided below:





Theorem

Remainder Estimate for the Integral Test:
Suppose $f(k) = a_k$, where f is continuous, positive, and decreasing on $[n, \infty)$ and $\sum a_n$ is a convergent series. Then

$$\int_{n+1}^{\infty} f(x)\ dx \leq R_n \leq \int_{n}^{\infty} f(x)\ dx$$

Example 1. Let $s = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

a. Approximate s using the first 10 terms.

b. Estimate the error in using this approximation.

c. How many terms are required to ensure that the sum is accurate within 0.0001.

d. Use Desmos to answer part c from above.