SERIES

MTH 253 LECTURE NOTES

Exploration: Writing an integer in base 10 means summing scalar multiple powers of 10 together. Writing a terminating or non-terminating decimal means the same thing.

$$8675309 = 8 \times 10^6 + 6 \times 10^5 + 7 \times 10^4 + 5 \times 10^3 + 3 \times 10^2 + 0 \times 10^1 + 9 \times 10^0$$

$$e = 2 + \frac{7}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{2}{10^4} + \frac{8}{10^5} + \frac{1}{10^6} + \cdots$$

A *sequence* is a bunch of values *listed* in order; a *series* is a bunch of values *added* in order.

Definition:

A **series** is the sum of the terms of a sequence. If $\{a_n\}$ is a sequence, then $\sum a_n$ or $\sum_{n=1}^{\infty} a_n$ is its associated series.

Some series have finite sums (the two above are finite – we put an equals sign between the series on the right and its sum on the left), and some are definitely infinite! For example,

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots + n + \cdots$$

Exploration: For some series, it isn't obvious whether they have a finite sum or not. In order to determine if the series sums to a value, we utilize sequences!

Definition:

The n**th partial sum** of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

If the sequence of partial sums, $\{s_n\}$, converges to a value s, then the series $\sum a_n$ is called **convergent**. In this case, we say that s is the **sum** of the series, and write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s$$

If the sequence of partial sums, $\{s_n\}$, diverges, then the series $\sum a_n$ is called **divergent**.

Note: Given a series $\sum_{n=1}^{\infty} a_n$, notice the difference between the nth partial sum of the series and the sum of the series.

$$n\text{th Partial Sum:} \qquad s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

$$\text{Sum:} \qquad s = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

Moreover, it is important to note that the sum of a series is the limit of the sequence of partial sums. That is,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Example 1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

a. Find the first 4 partial sums of the series.

b. Find a general formula for s_n.

c. Determine whether the series is convergent or divergent. If convergent, find its sum.

d. Use Desmos to graph the sequence, $\{a_n\}$, and the sequence of partial sums, $\{s_n\}$.

Exercise 1. Suppose the sum of the first n terms of $\sum a_n$ is

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{2n}{3n+5}$$

Determine if the series $\sum a_n$ is convergent or divergent. If it is convergent, find its sum.

Example 2. Consider the telescoping series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

a. Use a partial fraction decomposition to expand $\dfrac{1}{n(n+1)}$.

b. Write out the first 4 partial sums of the series, then find a general formula for s_n. (Note: This is why this series is called "telescoping".)

c. Determine whether the series is convergent or divergent. If convergent, find its sum.

d. Use Desmos to graph the sequence, $\{a_n\}$, and the sequence of partial sums, $\{s_n\}$.

Exploration: We are often able to reduce finding the sum of a series to finding the limit of the nth partial sum of the series. This is not always possible, but even when it is, it often takes some finesse.

Definition:

A **geometric series** is the sum of a geometric sequence.

Theorem:

The geometric series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$

- converges to $\dfrac{a}{1-r}$ when $|r| < 1$
- diverges when $|r| \geq 1$.

Proof:

Example 3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n$ converges or diverges. If it converges, find its sum.

Exercise 2. Determine whether the series $-6+5-\frac{25}{6}+\frac{125}{36}-\cdots$ converges or diverges. If it converges, find its sum.

Definition:

The series $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ is called the **Harmonic Series**.

Exploration: Show that the harmonic series is a divergent series.

Note: A geometric series with $r>1$ is a quick test to show a series is definitively divergent, since its terms continue to grow. Another test follows.

Theorem:

If $\sum_{n=1}^{\infty}a_n$ is convergent, then $\lim_{n\to\infty}a_n=0$.

Theorem:

The Divergence Test

If either $\lim_{n\to\infty}a_n$ DNE, or $\lim_{n\to\infty}a_n\neq 0$, then $\sum_{n=1}^{\infty}a_n$ diverges.

Note: The converse is not true! Just because $\lim_{n\to\infty}a_n=0$ does not imply that $\sum_{n=1}^{\infty}a_n$ converges. The Harmonic Series is a perfect example of this.

Exercise 3. Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ converges or diverges. If the series converges, find its sum.

Theorem:

If $\sum a_n$ and $\sum b_n$ each converge, then

- $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
- $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

Example 4. Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{3}{n^2+n} + \frac{1}{2^n}\right)$ converges or diverges. If the series converges, find its sum.

Exercise 4. Use a table in Desmos to list the sequence and sequence of partial sums for the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ up to n = 10. Use this to decide whether the series converges or diverges. If the series appears to converge, guess its sum.