# DIRECTION FIELDS

MTH 253 LECTURE NOTES

As we've seen so far, many real-world situations can be modeled by differential equations. Unfortunately, most differential equations are quite challenging, or oftentimes impossible, to solve; that is, to obtain an explicit formula for the solution. As an alternative, we can use something called direction fields (or slope fields) to graph the family of solutions or the specific solution to an initial value problem, allowing us to still learn a lot about the solution.

**Definition**

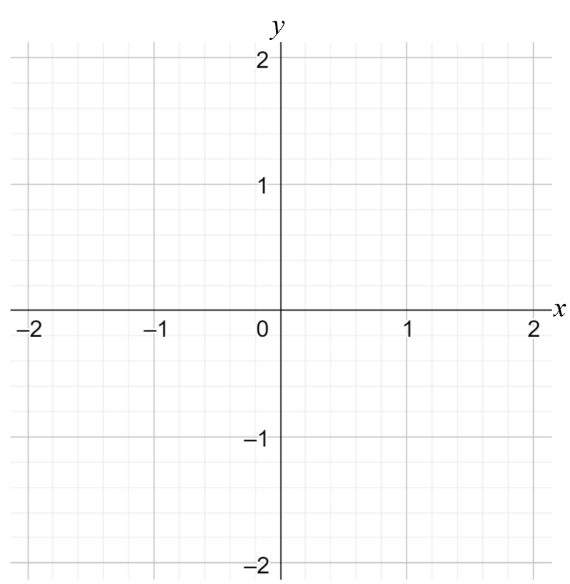
A **direction field** (or **slope field**) is a graphical representation of the slope of the solutions to a differential equation.

We will focus on first order differential equations. To create a direction field, we consider various $x$- and $y$-values and calculate the slope of the solution to the differential equation at that coordinate point $(x, y)$. We then draw a short line segment at $(x, y)$ with that particular slope. We continue to do so until we have filled up our coordinate system and then we can visualize the family of solutions to the differential equation through this direction field.

**Example 1.** Consider the differential equation $\frac{dy}{dx} = x + y$.

a) Fill in the following table of values.
b) Use the table of values to sketch a direction field for the differential equation.
c) Sketch the solution curve through $(0, 1)$. (This is a graphical representation of the solution to the IVP.)

| $x$ | $y$ | $\frac{dy}{dx} = x + y$ | $x$ | $y$ | $\frac{dy}{dx} = x + y$ |
|---|---|---|---|---|---|
| $-2$ | $-2$ | | $0$ | $1$ | |
| $-2$ | $-1$ | | $0$ | $2$ | |
| $-2$ | $0$ | | $1$ | $-2$ | |
| $-2$ | $1$ | | $1$ | $-1$ | |
| $-2$ | $2$ | | $1$ | $0$ | |
| $-1$ | $-2$ | | $1$ | $1$ | |
| $-1$ | $-1$ | | $1$ | $2$ | |
| $-1$ | $0$ | | $2$ | $-2$ | |
| $-1$ | $1$ | | $2$ | $-1$ | |
| $-1$ | $2$ | | $2$ | $0$ | |
| $0$ | $-2$ | | $2$ | $1$ | |
| $0$ | $-1$ | | $2$ | $2$ | |
| $0$ | $0$ | | | | |



**Technology Exploration:** Plotting Direction Fields and Solutions in GeoGebra
You can plot a direction field in GeoGebra Classic using the `slopefield` command and can sketch the solution to an initial value problem using the `solveODE` command. Let's do so to check our answers to the previous example.

**Exploration:** Recall the differential equation $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$. We showed that all functions of the form $y = \frac{cx^2}{1-cx}$ are solutions to that differential equation. We further found that the solution to the initial value problem where $y(2) = 1$ was $y = \frac{x^2}{6-x}$. Use GeoGebra to graph the slope field, then visualize the family of functions $y = \frac{cx^2}{1-cx}$ using a slider for $c$ and confirm the solution to the initial value problem.

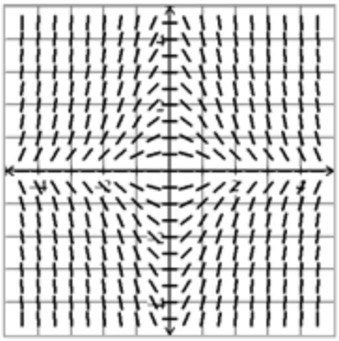
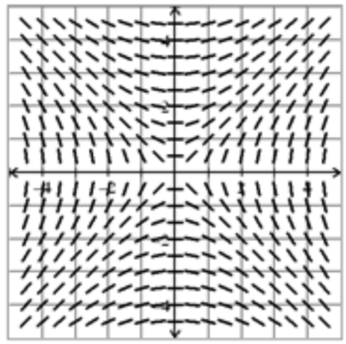
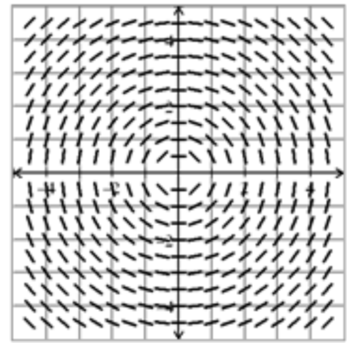
**Example 2.** Consider the differential equation $\frac{dy}{dx} = x^2 - y$.

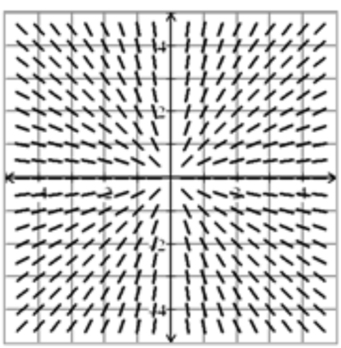
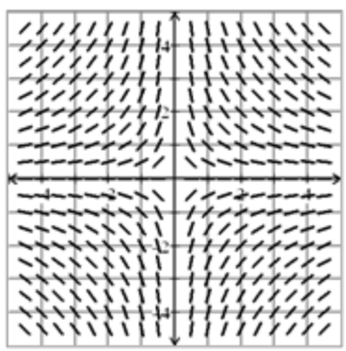
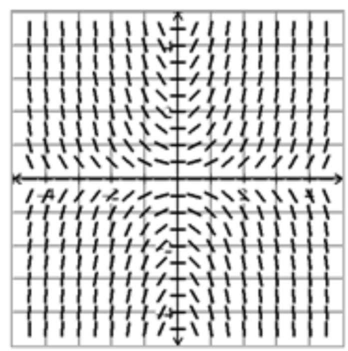
a) Use GeoGebra to sketch the direction field for the differential equation.

b) Sketch the solution curve that passes through $(-4, -8)$ using the Pen tool found under the Move menu.

c) Check your solution using the `solveODE` command.

**Exercise 1.** Without using technology, determine which of the following might be the slope field for the differential equation $\frac{dy}{dx} = \frac{y}{x}$.

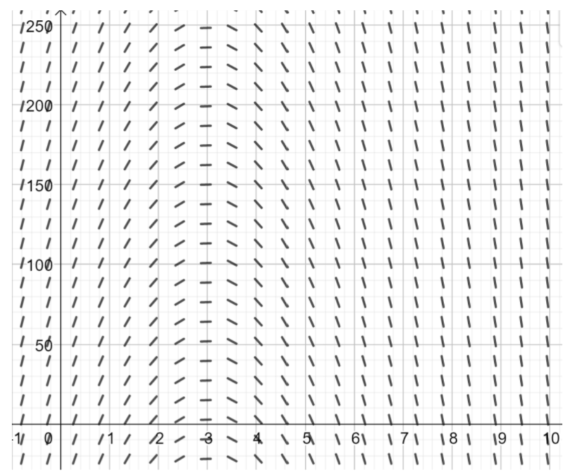




**Example 3.** Consider an object that has been launched vertically into the air with an initial velocity of 96 ft/sec. The solution to the differential equation $\frac{ds}{dt} = 96 - 32t$ gives the position $s$, in feet, of the object at any moment in time, $t$ in sec. The direction field for this differential equation can be seen below.

a) When will the object reach its maximum height? Find this using both the differential equation and its direction field.

b) Sketch the solution to the initial value problem if $s(0) = 0$.

c) Check your answer to the sketch from above by finding the explicit equation of $s(t)$ if $s(0) = 0$.




**Exercise 2.** Consider the differential equation $\frac{dy}{dx} = (y+2)(y-3)$.

a) Show that $y = -2$ and $y = 3$ are equilibrium solutions to the differential equation.

b) The direction field of the differential equation is provided. Sketch the equilibrium solutions along with the solution curve that passes through the origin.