MODELING WITH DIFFERENTIAL EQUATIONS

MTH 253 LECTURE NOTES

Definition

A **differential equation** (often pronounced "diff-E-Q" or "DE" for short) is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in that equation. A **solution** to a differential equation is a function (or family of functions) that satisfies the equation. That is, a function f is a **solution** if the differential equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

Note: The notations $\frac{dy}{dx}$ or y' can be used interchangeably. You will oftentimes now see y' used instead of $\frac{dy}{dx}$ due to its convenience to write.

Example 1. State the order of each differential equation and see if you can mentally solve each one using what you've learned so far in Calculus (essentially "thinking backwards").

a. $\frac{dy}{dx} = x$ b. $\frac{dy}{dx} = y$ c. $y'' = -y$

Exercise 1. Which of the following is a solution to the differential equation $y'' + 4y = 0$?

a. $y = e^{-2x}$ b. $y = \sin(2x)$

Definition

An **Initial Condition** for a differential equation is a statement of given values for the dependent and independent variables of the solution (often in the form of a point). An **Initial Value Problem (IVP)** is a differential equation along with an initial condition.

Example 2. Consider the differential equation $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$.

a. Show that all functions of the form $y = \frac{cx^2}{1-cx}$ are solutions to the differential equation.

b. Solve the initial value problem if $y(2) = 1$ (that is, find c).

Exploration: Modeling helps us better understand the world around us and predict future events. For example, if we toss an object vertically into the air, we know that the acceleration due to gravity will cause the object to return to the earth. Furthermore, we can use modeling to predict both the position and velocity of the object at any moment in time using

$$s(t) = s_0 + v_0 t - \frac{a}{2}t^2$$
$$v(t) = v_0 - at$$

where s_0 is the initial position, v_0 is the initial velocity, and a is the acceleration due to gravity. Modeling other scenarios, such as climate change, population, the spread of a virus, etc. are also important endeavors. Oftentimes in real-world situations, we notice that changes occur and we want to predict future behavior based on how current values are changing. These relationships are often best represented by a differential equation.

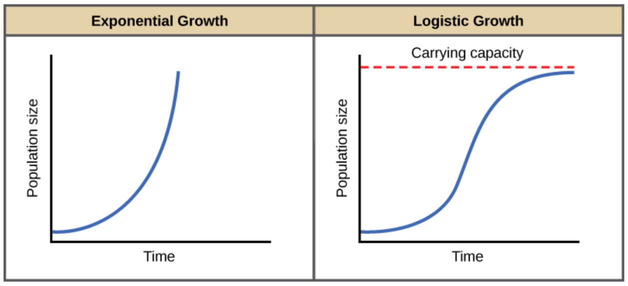
Example 3. The rate of growth of a population, $\frac{dP}{dt}$, is proportional to the population size, P. This can be written as a differential equation

$$\frac{dP}{dt} = kP$$

a. Verify that $P = Ce^{kt}$ is a solution to the differential equation.

b. Visualize this solution by graphing a subset of the family of solutions (pick a specific k-value and then graph for various C-values).

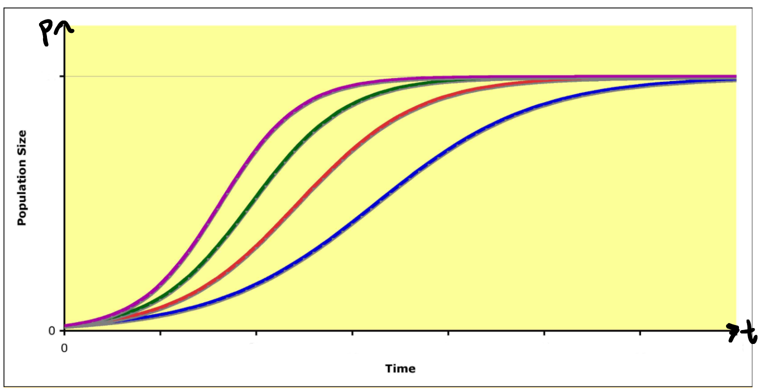
Note: It is worth noting that k is a given constant but C is an arbitrary constant. The constant k represents the rate of growth (note that k is present in both the differential equation and the solution to the differential equation) and C represents the initial population.



The exponential growth model assumed that the environment is unlimited, nutrition is available, there are no predators nor diseases, etc. The population can continue to grow uninhibited. This is oftentimes not the case though, and population usually levels off when it approaches its carrying capacity. For a model to take into account both exponential growth and carrying capacity, we often use the *logistic differential equation* below, where P is the population and M is the carrying capacity. This differential equation was proposed by the Dutch mathematical biologist Pierre-Francois Verhulst in the 1840s as a model for world population growth.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Example 4. Consider the family of solutions to the logistic equation for various values of k shown below.



a. Label the figure with M, the carrying capacity.

b. Which solutions represent a larger value of k and which ones represent a smaller value of k?

Exercise 2. Let's make sense of the logistic equation by answering the following questions.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

a. If P, the population, is small compared to M, the carrying capacity, then what happens to the logistic equation? How does this make sense in the context of population?

b. If P, the population, is larger than M, the carrying capacity, then what happens to the logistic equation? How does this make sense in the context of population?

c. If P, the population, approaches M, the carrying capacity, then what happens to the logistic equation? How does this make sense in the context of population?

d. Show that the constant functions $P = M$ and $P = 0$ are both solutions to the logistic equation. How does this make sense in the context of population? (Note: These are called *equilibrium solutions*.)

e. Label the equilibrium solutions on the figure from the previous page.