NEWTON'S METHOD

MTH 253 LECTURE NOTES

Exploration: Solving equations is one of the most important things we do in mathematics. In algebra, you learned how to solve linear and quadratic equations. In precalculus, you learned how to solve rational, exponential, logarithmic, and trigonometric equations. Until now, these have involved "nice" equations that can be solved by certain means.

Exercise 1. Solve the equation $x^3 - x^2 = 0$.

We are surprisingly limited, however, in what we can solve analytically. For instance, changing the equation from above just slightly to $x^3 - x^2 - 1 = 0$, or the equation $\cos x = x$, cannot be solved by algebraic methods in terms of familiar functions. Fortunately, there are methods that can give us approximate solutions to equations like these, and these methods can usually give an approximation correct to as many decimal places as we like.

We will focus on a technique called Newton's Method, which is built around tangent lines. The idea is that if x is sufficiently close to a zero of $f(x)$, then the tangent line to the graph at $(x, f(x))$ will cross the x-axis at a point *closer to the zero* than the original x. This method takes advantage of the fact that it is quite easy to solve a linear equation.

Newton's Method, Geometrically

1. Begin with an initial guess roughly where the zero is – call this x_1.
2. Draw the tangent line to the graph at $(x_1, f(x_1))$. Follow this line to where it meets the x-axis. Call this x-intercept x_2.
3. Repeat, drawing the tangent line to the graph at $(x_2, f(x_2))$ to see where *it* meets the x-axis. Call this x-intercept x_3.
4. Repeat until sufficiently close.

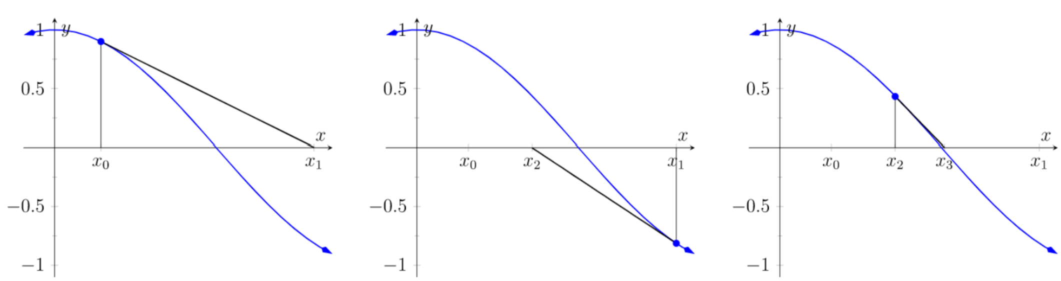


FIGURE 1. Newton's Method, via APEX Calculus, Newton's Method Section

Newton's Method, Algebraically

1. Begin with an initial guess roughly where the zero is – call this x_1.
2. Write the linearization $L_1(x)$ of f at x_1:

3. Find the x-intercept of $L_1(x)$. Call it x_2.

4. Repeat to find x_3, the x-intercept of $L_2(x)$.

5. Generally, an approximation x_n allows us to find the next approximation x_{n+1}:

Theorem

Newton's Method
Let f be a differentiable function on an interval I with a zero in I. To approximate the value of the zero, accurate to d decimal places,

1. Choose a value x_1 as an initial approximation of the zero (we often use a graph to aid in this).
2. Create successive approximations iteratively; given an approximation x_n, compute the next approximation x_{n+1} as
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
3. Stop the iterations when successive approximations do not differ in the first d places after the decimal point.

Technology Exploration: A GeoGebra Applet for Dynamic Visualization of Newton's Method.

Example 1. Consider the equation $x^3 - x^2 - 1 = 0$.

a. Starting with $x_1 = 1$, find the third approximation x_3 to the root of the equation.

b. Starting with $x_1 = 1$, find the root of the equation accurate to eight decimal places. How many iterations of Newton's Method were needed?

c. Starting with $x_1 = 1.5$, find the root of the equation accurate to eight decimal places. How many iterations of Newton's Method were needed?

Example 2. Find, correct to six decimal places, the root of the equation $\cos x = x$.

Exploration: Newton's Method may fail for several reasons, typically when the initial guess x_1 is not chosen well.

1 : **Zero Slope**. If $f'(x_1) = 0$, then we are stuck and cannot continue the process because the tangent line will have no x-intercept. For example, if we chose $x_1 = 0$ in the $x^3 - x^2 - 1 = 0$ example, then we would see the following

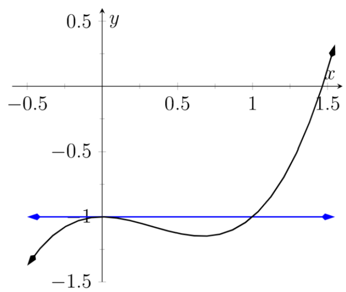


FIGURE 2. Image via Active Calculus

Furthermore, if $f'(x_1) = 0$, then $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ has a division by zero.

2 : **Near-Zero Slope**. Even if x_1 results in just *near* a zero slope, it can cause x_2 to be a worse approximation than x_1. For example, if $x_1 = 0.5$ in the above example, then the zero to our linearization will be much further away from our initial choice.

3 : **Non-differentiable**. If f is not differentiable at its x-intercept (such as a vertical tangent line), then Newton's Method will continue to fluctuate and not converge at the zero. Consider $f(x) = \sqrt[3]{x}$, for example.

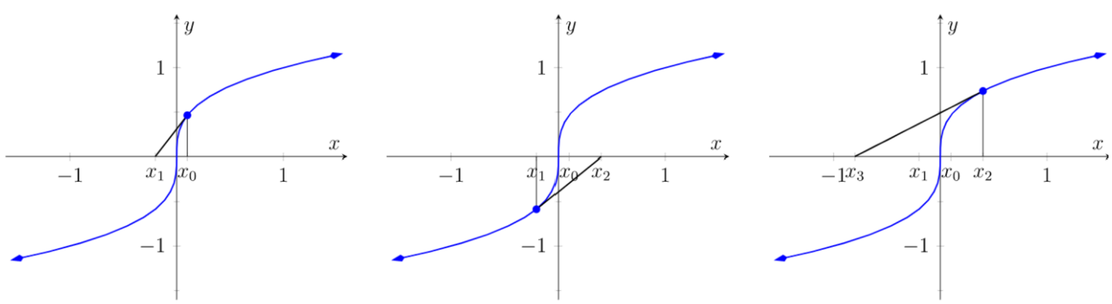


FIGURE 3. Image via Active Calculus

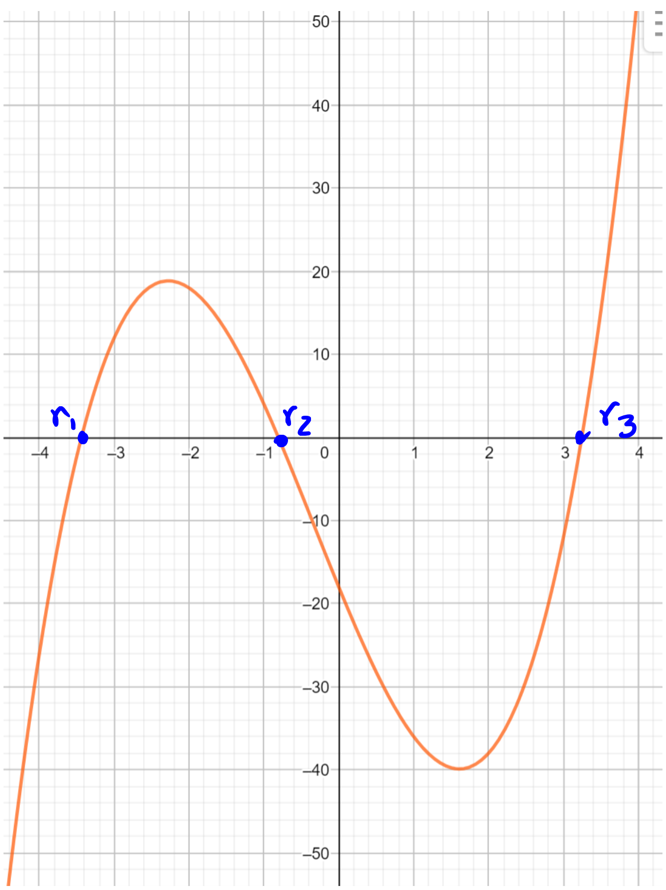
4 : **Domain Restrictions**. If the domain of f is not $\mathbb{R}$, then Newton's Method could result in the x-intercept of the linearization being outside of the domain of f. Consider $f(x) = \sqrt{x}$, for example.

Example 3. Use Newton's method to find all the roots of the equation accurate to eight decimal places.

$$x^3 - 2x^2 - 11x + 12 = -x^3 - 4x^2 + 11x + 30$$

$$2x^3 + 2x^2 - 22x - 18 = 0$$

Let $f(x) = 2x^3 + 2x^2 - 22x - 18$



Find r_1. Start with $x_1 = -4$.

	A
1	-4
2	-3.5517241379
3	-3.4443128996
4	-3.4380899382
5	-3.4380693994
6	-3.4380693992
7	-3.4380693992

$\leftarrow r_1 \approx -3.43806940$

Find r_2. Start with $x_1 = -1$.

	A
1	-1
2	-0.8
3	-0.806741573
4	-0.8067475701
5	-0.8067475701
6	-0.8067475701

$\leftarrow r_2 \approx -0.80674757$

Find r_3. Start with $x_1 = 3$.

	A
1	3
2	3.2727272727
3	3.2451206602
4	3.2448170059
5	3.2448169693
6	3.2448169693
7	3.2448169693

$\leftarrow r_3 \approx 3.24481700$