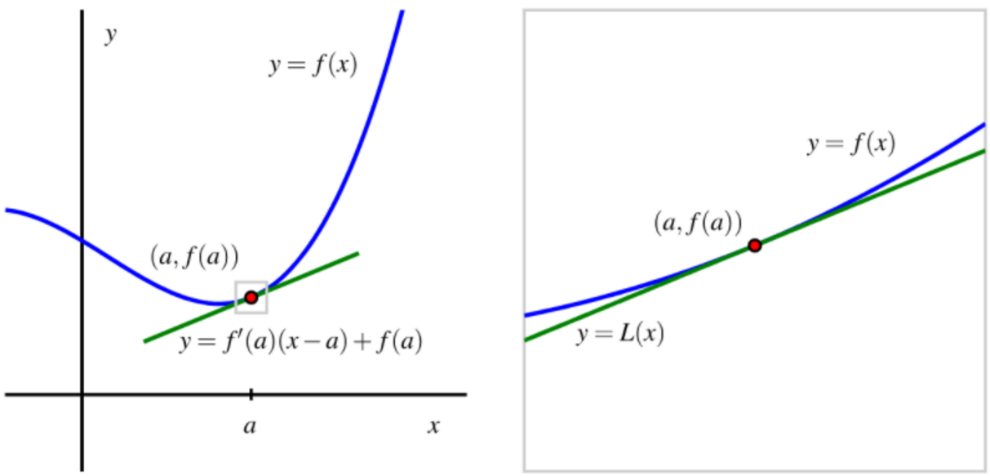
LINEAR APPROXIMATIONS & DIFFERENTIALS

MTH 253 LECTURE NOTES

Exploration: Given the graph of a function, if we zoom in on a point, the graph of the curve and the graph of its tangent line at that point will be nearly indistinguishable.



Definition

The **Linear Approximation** of f at a is

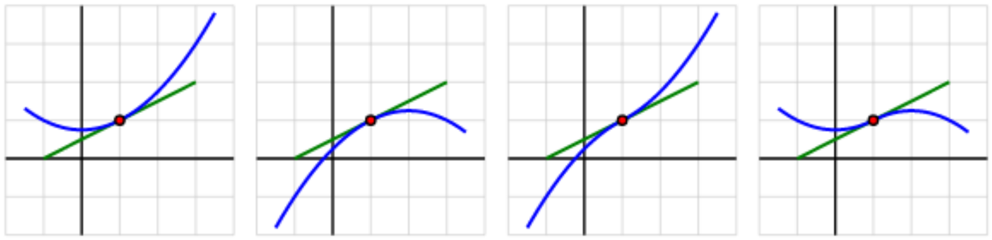
$$f(x) \approx f(a) + f'(a)(x-a)$$

The function $L(x)$ whose graph is the tangent line to f at a is called the **Linearization of f at a** and is

$$L(x) = f(a) + f'(a)(x-a)$$

Note: There is a major difference between $f(a)$ and $f(x)$ in this context. Notice that $f(a)$ is a constant that results from using the given value of a, while $f(x)$ is the general expression for the rule that defines the function. The same is true for $f'(a)$ and $f'(x)$. We must carefully distinguish between the two. Each time we find the tangent line, we need to evaluate the function and its derivative at a.

Overestimates & Underestimates



- When the curve is concave down at $x = a$, the linear approximation will produce an overestimate.
- When the curve is concave up at $x = a$, the linear approximation will produce an underestimate.
- When the curve has an inflection point at $x = a$, the linear approximation may produce an underestimate and overestimate depending on whether we are left or right of $x = a$.

Example 1. Consider the function $f(x) = \sqrt{x}$.

a. Find the linearization of f at $a = 4$.

b. Use the linearization found above to approximate the values of $\sqrt{5}$, $\sqrt{4.1}$, and $\sqrt{3.9}$. Are these approximations overestimates or underestimates?

c. Suppose we try to use the linearization above to approximate the value of $\sqrt{37}$. What would be a better approach to approximating $\sqrt{37}$ using linearization?

Exercise 1. Consider the function $f(x) = \sin x$.

a. Find the linearization of f at $a = 0$.
b. Use the linearization found above to approximate $\sin(0.1), \sin(0.5)$, and $\sin(1)$.
c. Use the linearization to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$. Confirm your result using L'Hôpital's Rule.

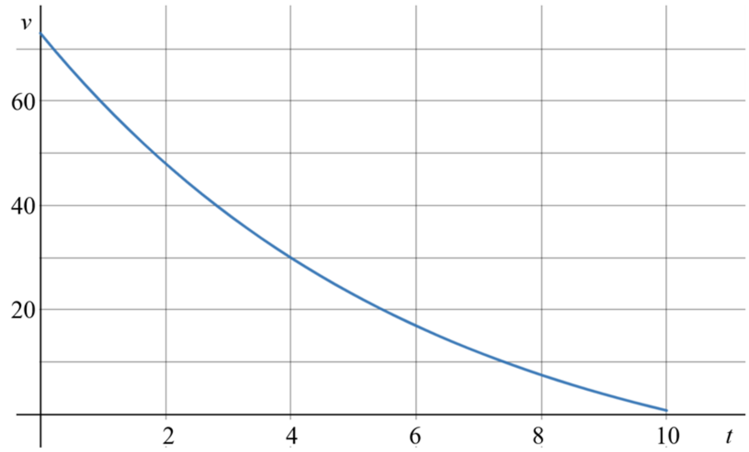
Provided a linear approximation, it is natural to wonder how accurate such an approximation is.

Example 2. Using technology, for what values of x is the linear approximation

$$\sqrt{x} \approx \frac{1}{4}x + 1$$

accurate to within 0.5? What about accuracy to within 0.1?

Example 3. The graph shown represents the velocity, v in ft/sec, of a decelerating car after t seconds. If it is known that the position of the car at time 4 seconds is 10ft, use a linearization to approximate the position of the car at time 4.5 seconds.



The ideas of linear approximations are often translated into other terminologies and notations. Differentials is one of those translations.

Definition

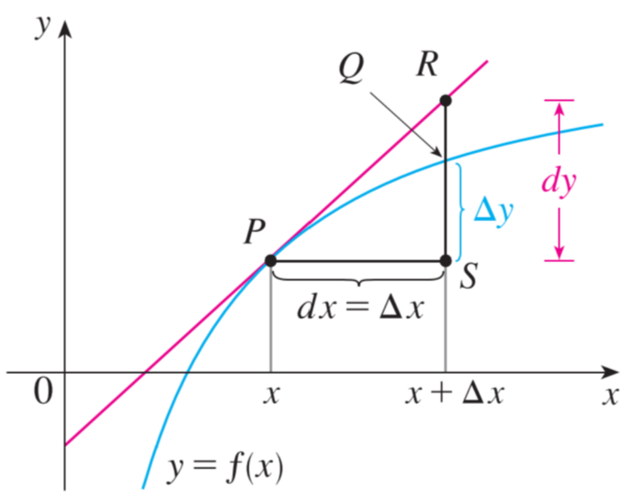
If $y = f(x)$ is differentiable, then the dx and dy are variables called **Differentials**, adhering to the equation

$$dy = f'(x)\, dx$$

where dx is independent, dy is dependent upon dx.

Notation: This notation is new, but the ideas are not. Suppose $y = f(x)$ is differentiable.

- Δx represents a small, nonzero change in the x-value.
- dx also represents a small, nonzero change in the x-value.
- $dx = \Delta x$.
- Δy represents the corresponding change in the y-value as x changes by Δx. That is, $\Delta y = f(x + \Delta x) - f(x)$.
- $dy = f'(x)\, dx$, which represents an approximation of the change in the y-value as x changes by Δx.
- $dy \approx \Delta y$.
- Δy is often difficult to compute, whereas dy is often easier to compute.



Note: If we solve $dy = f'(x)\, dx$ for $f'(x)$, we get $f'(x) = \frac{dy}{dx}$. This equation represents that $f'(x)$ is equivalent to the ratio of differentials; this is *not* an alternate notation for the Leibniz notation $\frac{dy}{dx}$, even though it looks the same. The Leibniz notation was chosen because of the fraction-like qualities of the derivative but is truly a *symbol*, not a fraction.

Example 4. Consider the function $f(x) = \ln(x^2)$.

a. Find the differential dy.
b. Evaluate dy and Δy if $x = 1$ and $dx = 0.1$.
c. Evaluate dy and Δy if $x = 1$ and $dx = 4$. Explain the discrepancy.