13.1 Vector Fields

13.1.1 An Introduction to Vector Fields

Definition

Let D be a set in $\mathbb{R}^n$. A **scalar field on** $\mathbb{R}^n$ (or **scalar function**) is a function f that assigns to each point $(x_1, x_2, \ldots, x_n) \in D$ a scalar $f(x_1, \ldots, x_n) \in \mathbb{R}$. That is, a scalar field is a function $f : D \to \mathbb{R}$ by $(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$.

Some examples of scalar fields would be

- $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.
- $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) = x^2 - y^3$.
- $h : \mathbb{R}^3 \to \mathbb{R}$ by $h(x, y, z) = xyz + 1$.
- $T : \mathbb{R}^3 \to \mathbb{R}$, where $T(x, y, z)$ is the temperature in degrees Fahrenheit at every point in the room you are in.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^2 - y^3$. Then f is a scalar field on $\mathbb{R}^2$.

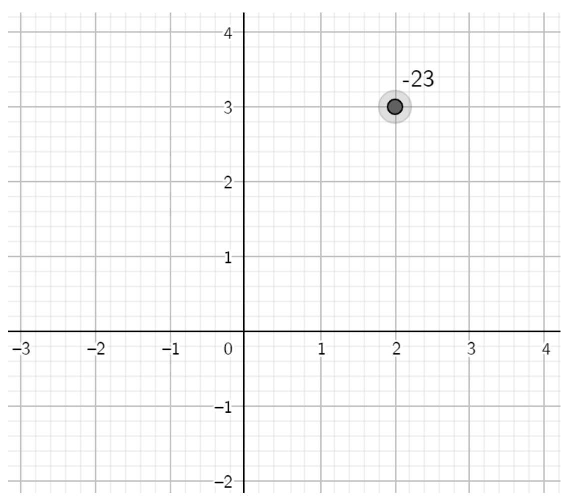


Figure 1: A Scalar Field
https://www.geogebra.org/calculator/eqccnshr

We introduce a scalar field to differentiate it from a vector field.

Definition

Let D be a set in $\mathbb{R}^2$. A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point $(x, y) \in D$ a two-dimensional vector $\mathbf{F}(x, y)$. That is, a vector field is a function $\mathbf{F} : D \to \mathbb{R}^2$ by $(x, y) \mapsto \mathbf{F}(x, y)$.
If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$, then the scalar functions P and Q are called **component functions** for $\mathbf{F}$.

Definition

Let $E \in \mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F} : E \to \mathbb{R}^3$ by $(x, y, z) \mapsto \mathbf{F}(x, y, z)$.

Convention

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ with component functions P, Q, R, then we may write either

$$\begin{aligned} \mathbf{F}(x, y, z) &= P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} \\ \mathbf{F}(\mathbf{x}) &= P(\mathbf{x})\mathbf{i} + Q(\mathbf{x})\mathbf{j} + R(\mathbf{x})\mathbf{k} \\ \mathbf{F} &= P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \end{aligned}$$

where $\mathbf{x}$ is understood to be the position vector for (x, y, z).

Suppose $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ by $P(x, y) = \ln|x| - \cos x$ and $Q(x, y) = \sqrt{|y|} - y \sin y$. Then $\mathbf{F}$ is a vector field on $\mathbb{R}^2$.

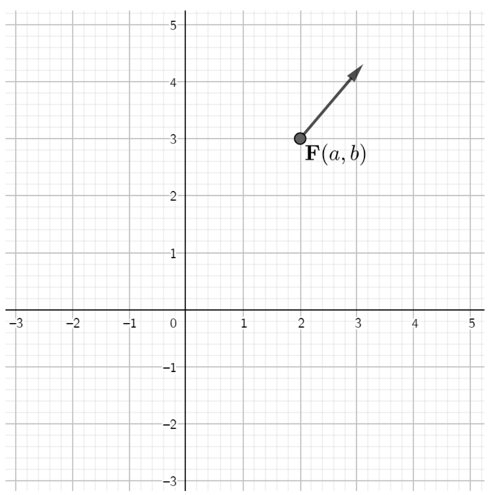
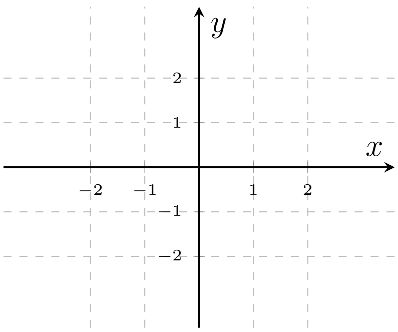


Figure 2: A Vector Field
https://www.geogebra.org/calculator/tswxsams

Example 1. Let $\mathbf{F}(x, y) = (x+y)\mathbf{i}+(x-y)\mathbf{j}$ be a vector field on $\mathbb{R}^2$. Describe $\mathbf{F}$ by sketching several vectors $\mathbf{F}(x, y)$ on $[-2, 2] \times [-2, 2]$.



Technology: For nice examples of vector field visualizations, check out these applets:

- Vector Fields on $\mathbb{R}^2$: https://www.geogebra.org/m/QPE4PaDZ
- Vector Fields on $\mathbb{R}^3$: https://www.geogebra.org/m/u3xregNW

13.1.2 Conservative Vector Fields

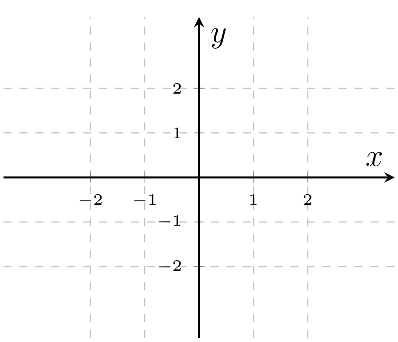
Definition

If f is a scalar function on $\mathbb{R}^2$ or $\mathbb{R}^3$, then the gradient ∇f is actually a vector field. That is,

$$\begin{aligned} \nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \langle f_x(x, y), f_y(x, y)\rangle \\ \nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle \\ \nabla f &= f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} = \langle f_x, f_y, f_z\rangle \end{aligned}$$

and ∇f is called a **gradient vector field**.

Example 2. Sketch the gradient vector field of $f(x, y) = \sqrt{x^2 + y^2}$.



Definition

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient vector field of some scalar function. That is, $\mathbf{F}$ is conservative if there exists some f such that $\mathbf{F} = \nabla f$. In this case, we call f a **potential function** for $\mathbf{F}$.

Note: Note the parallelism.

- F is integrable iff there exists f such that $f' = F$.
- $\mathbf{F}$ is a conservative vector field iff there exists f such that $f' = \mathbf{F}$.
- f is an antiderivative of F iff $f' = F$.
- f is a potential function for $\mathbf{F}$ iff $\nabla f = \mathbf{F}$.