12.8 Triple Integrals in Cylindrical & Spherical Coordinates

12.8.1 Triple Integrals in Cylindrical Coordinates

Recall Cylindrical Coordinates: https://www.geogebra.org/3d/yesa2uja

Cylindrical → Rectangular	**Rectangular → Cylindrical**
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = z$	$z = z$

Theorem

Suppose $\mathcal{W}$ is a Type I solid region whose projection $\mathscr{D}$ onto the xy-plane has polar description

$$\mathscr{D} = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

Then

$$\begin{aligned}
\iiint\limits_{\mathcal{W}} f(x,y,z)\, dV &= \iint\limits_{\mathscr{D}} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right) dA \\
&= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta \\
&= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{z_1(r,theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta
\end{aligned}$$

where $h_i(x,y) = u_i(r\cos\theta, r\sin\theta) = z_i(r,\theta)$ are different expressions for the same relationship.

Example 1. Evaluate $\iiint\limits_{\mathcal{W}} z\sqrt{x^2+y^2}\ dV$, where W is the cylinder $r^2 \leq 4$ with $1 \leq z \leq 5$.

Example 2. Evaluate $\iiint\limits_{\mathcal{W}} z\ dV$, where $\mathcal{W}$ is the region within the cylinder $x^2+y^2 \leq 4$ with $0 \leq z \leq y$.

12.8.2 Triple Integrals in Spherical Coordinates

Recall Spherical Coordinates: https://www.geogebra.org/3d/qjk6hmgz

Spherical → Rectangular	Rectangular → Spherical
$x = \rho\sin\varphi\cos\theta$	$\rho^2 = x^2 + y^2 + z^2$
$y = \rho\sin\varphi\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = \rho\cos\varphi$	$\varphi = \arccos\dfrac{z}{\rho}$

Whereas with double integration in polar coordinates we integrated over a polar rectangle, here we triple integrate in spherical coordinates over a spherical wedge.

Definition

A **spherical wedge** is a solid described in spherical coordinates by $\mathcal{W} = \underbrace{[a,b]}_{\rho} \times \underbrace{[\alpha,\beta]}_{\theta} \times \underbrace{[\chi,\psi]}_{\varphi}$, where $a \geq 0$, $0 \leq \beta - \alpha \leq 2\pi$, and $0 \leq \psi - \chi \geq \pi$.

If we generalize a spherical wedge to allow ρ to vary between to surfaces dependent upon θ and φ, then we get the following theorem.

Theorem

Suppose f is integrable over a region $\mathcal{W}$ defined by

$$\mathcal{W} = \{(\rho,\theta,\varphi) \mid \alpha \leq \theta \leq \beta, \chi \leq \varphi \leq \psi, \rho_1(\theta,\varphi) \leq \rho \leq \rho_2(\theta,\varphi)\}$$

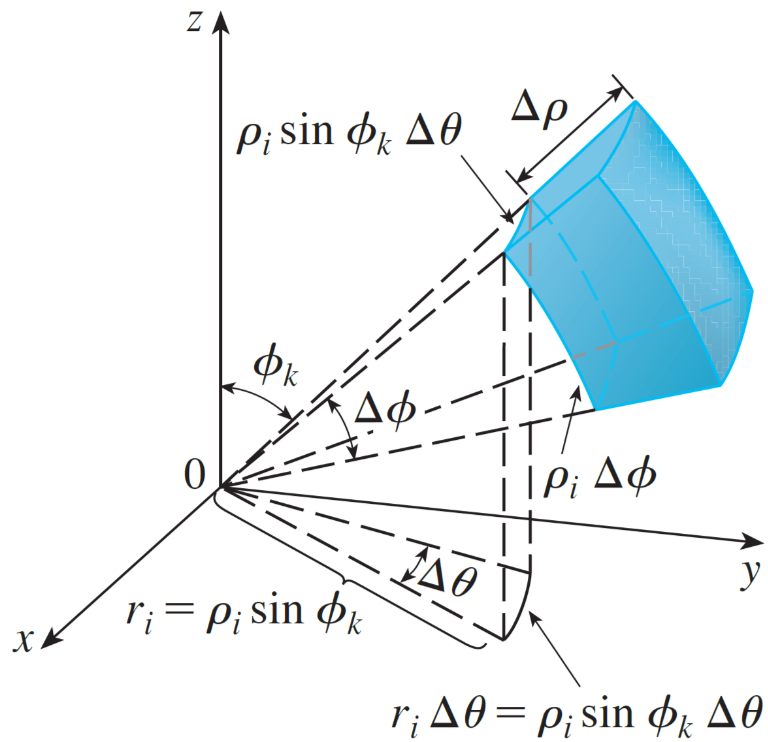
then

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV = \int_{\chi}^{\psi}\int_{\alpha}^{\beta}\int_{\rho_1(\theta,\varphi)}^{\rho_2(\theta,\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi d\rho\, d\theta\, d\varphi$$

A rationale is provided on the last page of this section.

Example 3. Compute the integral of $f(x, y, z) = x^2 + y^2$ over the sphere $\mathcal{S}$ of radius 3 centered at the origin.

Example 4. An ice cream filling a cone and making a spherical top can be approximately modeled according to the surfaces $z^2 = x^2 + y^2$ and $z = x^2 + y^2 + z^2$. It is known that the ice cream top is represented by a hemisphere, and two opposite edges make a right angle at the tip of the cone. Find the volume of the ice cream. (See https://www.geogebra.org/3d/vhqabgzk)



Spherical Wedge – Stewart, *Concepts & Contexts*, 4E, Figure 7, pp. 885

12.8.3 Rationale for the Triple Integrals Formula in Spherical Coordinates

Define a spherical wedge $\mathcal{W} = [a,b] \times [\alpha,\beta] \times [\chi,\psi]$, where $a \geq 0$, $0 \leq \beta - \alpha \leq 2\pi$, and $0 \leq \psi - \chi \geq \pi$. Divide $\mathcal{W}$ into subwedges $\mathcal{W}_{ijk}$ by cutting $\mathcal{W}$ with equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\varphi = \varphi_k$. Then $\mathcal{W}_{ijk}$ can be approximated by a rectangular box of dimensions $A \times B \times C$.

Now, $A = \Delta\rho$ (side extending away from the origin). For the side rotating according to θ, B, we have the length of an arc of a circle. The radius of that circle is r, and the angle of that arc is $\Delta\theta$. Converting r to spherical coordinates, we get that $r = \rho_i \sin\varphi_k$. Since the length of the arc of a circle is the product of the radius with the angle subtended by it, $B = \rho_i \sin\varphi_k \Delta\theta$. Lastly, we have the side rotating according to φ, C. For this side, we also have an arc of a circle whose radius is ρ_i, and the angle subtended by it is $\Delta\varphi$. It follows that $C = \rho_i \Delta\varphi$. Thus, an approximation for the volume of $\mathcal{W}_{ijk}$ is

$$\begin{aligned} \text{size}(\mathcal{W}) &\approx \Delta\rho \times \rho_i \Delta\varphi \times \rho_i \sin\varphi_k \Delta\theta \\ &= \rho_i^2 \sin\varphi_k \Delta\rho \ \Delta \ \Delta\theta \ \Delta\varphi \end{aligned}$$

We now consider the Riemann sum for $f(x,y,z)$. That is

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \ \Delta V$$

$$= \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho_i \sin\varphi_k \cos\theta_j, \rho_i \sin\varphi_k \sin\theta_j, \rho_i \cos\varphi_k) \rho_i^2 \sin\varphi_k \ \Delta\rho \ \Delta\theta \ \Delta\varphi$$

Taking limits as $\ell, m, n \to \infty$, we get

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\ dV$$

$$= \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho_i \sin\varphi_k \cos\theta_j, \rho_i \sin\varphi_k \sin\theta_j, \rho_i \cos\varphi_k)\rho_i^2 \sin\varphi_k\ \Delta\rho\ \Delta\theta\ \Delta\varphi$$

$$= \int_{\chi}^{\psi}\int_{\alpha}^{\beta}\int_{\rho_1(\theta,\varphi)}^{\rho_2(\theta,\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi d\rho\ d\theta\ d\varphi$$