The Properties & Applications of Fractal Geometry

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Abstract

Fractal shapes hold a set of odd properties which may seem like a simple peculiarity to the untrained eye. Fractal functions can create lines which have infinite length in a finite area, or 2-dimensional surfaces which can completely fill a 3-dimensional space, even stunning artwork such as the famous visual representations of the Julia or Mandelbrot sets. These properties can be utilized to model natural and artificial phenomena, such as the formation of frost crystals on a sheet of glass, or the length of the coastline of an island nation. No class of function tackles these applications better than fractals, and few, if any, hold quite as unique properties.

1 Introduction

The concept of fractal shapes was first introduced in the 17^{th} century by philosopher Gottfried Leibniz in his pondering of "recursive self similarity," and use of fractional exponents, a new concept to the field of geometry (Dzik, Vesely, & Zmeskal, 2013). Then, in 1872, Karl Weierstrass published what is now considered the first example of a fractal function (Turner 1998):

$$W(x) = \sum_{n=1}^{\infty} b^n cos(a^n \pi x)$$

This function represents an infinite sum of cosine terms, whose graph begins to look like an infinitely more jagged line as n approaches ∞. The curve created by this function also holds the strange property of being differentiable nowhere, yet continuous everywhere along the curve. While this property outraged many of Weierstrass' peers at the time, it later became one of the defining traits of fractal expressions (Turner 1998).

After Weierstrass' discovery, minor advances in understanding fractal geometry were made for roughly a century until the groundbreaking work of French mathematician Benoît B. Mandelbrot. Mandelbrot used fractal geometry to describe natural and artificial phenomena, providing considerably better models than had been possible previously. His work laid the foundation for research into fractal mathematics, philosophy, and applications, upon which the flourishing field of fractal geometry has grown. Mandelbrot famously summarized these applications in what he said became "an instant cliché": "Clouds are not spheres, mountains are not cones, coastlines are not circles and bark is not smooth, nor does lightning travel in a straight line" (Encyclopedia of World Biography 2005).

2 Mathematical Properties

2.1 Dimensionality

Fractal patterns are often characterized by their **fractal dimension**, a measure of how the complexity of a pattern changes as the scale at which it is examined changes. These dimensions work somewhat similarly to classical dimensions, with the main exception that they do not necessarily need to be integer values. For instance, a line segment would be a 1-dimensional shape, a square a 2-dimensional shape, a cube would be 3-dimensional, and so forth. However, a $1\frac{1}{2}$-dimensional shape is made possible using fractals, representing, in this example, something in-between a line and a square.

The first mathematicians to propose these "in-between" dimensions were Felix Hausdorff and Abram Besicovitch, resulting in a revolution in our understanding of geometry. Because of their discovery, we refer to such in-between dimensions as the **Hausdorff-Besicovitch dimension**, which may be calculated in a number of ways (Wahl 1995).

2.1.1 Standard Geometries

The basic equation used in calculating the dimension of an object relates its dimension D, to the number of pieces in the shape n, and the scale of each piece with respect to the whole s:

$$n = \frac{1}{s^D}$$

To illustrate this relationship, consult the following figure, which shows the relationship between n, s, & D in the formation of a standard lines, squares and cubes:

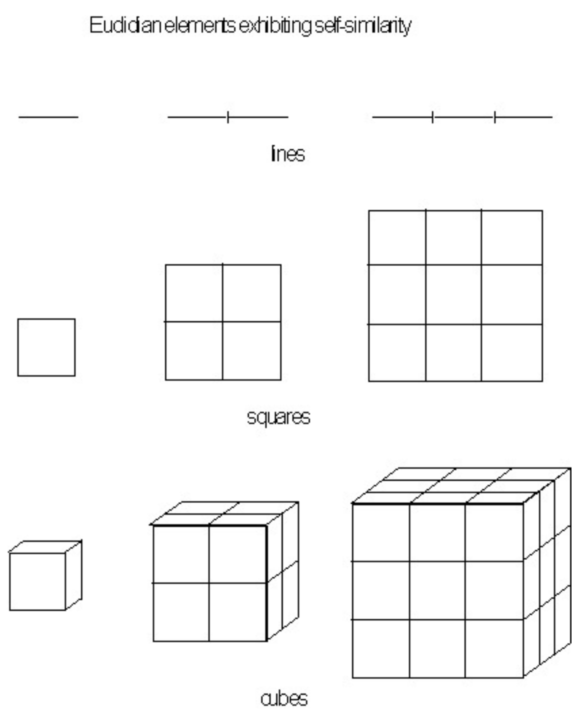

Figure 1: An example of the relation between n, s, and D. D increases toward the bottom of the figure, and s decreases to the right, causing n, being the number of pieces in the shape, to increase down and to the right (Wahl 1995).

The number of 'pieces' in each figure are as follows:

	$D=1$	$D=2$	$D=3$	$D=\ldots$
$s=1/1$	1	1	1	...
$s=1/2$	2	4	8	...
$s=1/3$	3	9	27	...
$s=$ ...	...	...	...	$n=\frac{1}{s^D}$

Figure 2: Table displaying the number of sub-pieces of each shape in figure 1 based on dimension and scale. A line with $s=\frac{1}{3}$ will contain 3 self-similar lines, a cube 27 self-similar cubes, so on and so forth (Wahl 1995).

2.1.2 Perfectly Self-Similar Fractal Geometry

Conveniently, perfectly self-similar fractal dimension calculations function very similarly, using the same formula, which may be solved for D and written as the following:

$$D=\frac{\log n}{\log \frac{1}{s}}$$

A perfectly self-similar fractal is any fractal which appears the exact same when examined on a small scale as it does on a larger scale, appearing perfectly similar to itself all the way down to an infinitely small scale. A popular example of this is the Koch curve, shown below, created by removing the middle $\frac{1}{3}$ of a line segment, and replacing it with two more line segments, forming a smaller equilateral triangle with the third side being the removed portion of the line segment. This process is then repeated for each of the smaller constituent segments indefinitely.

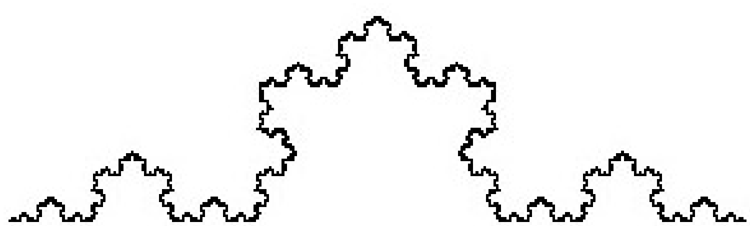

Figure 3: The Koch Curve, a popular example of perfectly self-similar fractal geometry (Yale Math).

Examining the Koch Curve, we can see that it is comprised of 4 perfectly self-similar versions of itself, meaning that its value for n in our formula is 4. We also know that our value for $s=\frac{1}{3}$ from our method of generating the curve, which scales it to $\frac{1}{3}$ of its previous size at each stage. Thus, using our formula for calculating dimension, we find that the Hausdorff-Besicovitch dimension for the Koch Curve is ~ 1.26. Now we have an idea of what an object in a fractional dimension might look like, in this case being somewhere between a 1-dimensional line segment, and a 2-dimensional square.

2.1.3 Non-Perfectly Self-Similar Fractal Geometry

Unfortunately, many fractals, such as coastlines or leaf structures, are not perfectly self-similar, rendering an exact calculation of their dimension impossible. However, an English meteorologist named Lewis Fry Richardson pioneered a method for approximating the dimension of these fractals. Richardson principally examined the boundaries of countries in his work, seeking to discover the reason behind inconsistencies in the measured lengths of the borders and coastlines of different countries. For example, at the time, "Spain claimed its boarder [sic] with Portugal was 987 km, whereas Portugal claimed it was 1214 km" (Wahl 1995). These discrepancies were widespread and created an administrative nightmare.

After careful study, Richardson proposed that the length of each measurement taken by different countries was to blame. A country which used a shorter "measuring stick" would come up with a larger number than if a longer one was used, as a shorter measurement would better fit to the irregularities of the border. To illustrate, if one was to measure a circle with three line segments, they would come up with a value much smaller than if one were to measure with four, or five, only arriving at the true value when an infinite value of infinitely small segments were to be used. The same concept applies to coastlines and borders, albeit to a much more complex degree.

Armed with the knowledge of his new discovery, Richardson then devised a way to approximate the dimension of a non-self-similar fractal, as well as the true perimeter of whatever non-self-similar fractal shape he was dealing with.

The perimeter calculation is quickest. One simply measures the object using units of decreasing size, graphs them, and finds an equation of best fit. Then take the limit of the equation, and the true perimeter may be estimated by the limit.

But what we care about is Richardson's method for calculating their dimension. At first, the method is the exact same. One measures the shape using units of decreasing size, but then, the perimeters collected are graphed on a graph of $log(\frac{1}{s})$ vs $log(n)$, in order to create a graph whose line of best fit will be a straight line with constant slope m between 0 and 1. Then, the dimension of the shape may be found by the equation $m + 1 = D$. Completing this operation for the Koch Curve yields a similar approximation to its exact calculation $D \sim 1.26$, confirming its relative accuracy. Unfortunately, this method is limited by its requirement of using instruments to collect data. This limits the calculation to only applying to objects in dimensions which we can measure. For instance, it could not be utilized to measure a fractal of dimension greater than four, because we have access to no "Four-dimensional measuring sticks" (Wahl 1995). Though, generally speaking, the fractal dimension of a non-perfectly self-similar shape in more than four-dimensional is not particularly applicable anyway.

2.2 Unintuitive Behaviors

As one can imagine, such unique mathematical objects also hold a set of equally unique, and strange properties. Some are useful, others not as much. Some may simply remain in the realm of 'cool.' Whatever the case, these properties are worth exploring, potentially leading us to new discoveries, optimizations, and ultimately bettering the human experience.

2.2.1 Fractional Dimensionality

One such property, which we have already explored computationally, is the property of fractional dimensions. These dimensions, in a more conceptual sense, are a measure of the amount of information the particular curve contains (Wahl 1995). For example, remember the Koch Curve discussed earlier, with a dimension of roughly 1.26, and a standard 1-dimensional line and 2-dimensional plane.

Now, think of them in terms of the amount of information one could represent with them. The line could hold an infinite amount of information of any one thing (such as the number line most learned in early grade school), whereas the plane could hold the same for two (such as the Cartesian plane we often work with when graphing). However, if we were to try to store information on a Koch Curve, somewhat like a strange number line, we would be able to represent an infinite amount of information for any one thing, represented by the 1, and a compressed semi-infinite amount of information on a second thing (represented by the .26). We obviously cannot do this well by hand, but computer programs are capable of it, and some use it in order to compress the large volumes of information they must store.

2.2.2 Infinite Resolution

All fractals also hold the property of infinite resolution. That is, they have no hard, defined edges, and if one were to 'zoom in' on any point on the fractal, they would continue to find an increasingly complex shape. The shape of a fractal is therefore 'infinitely complex' (Mathigon).

2.2.3 Self-Similarity

Also touched on earlier is the fractal property of self-similarity. Fractals are generally formed from some kind of constant base, and an additional recurring generating process. Because this process is inherently recursive, it follows that the shape produced by it will express self-similarity. A popular, yet complicated example of this is the Mandelbrot set. To illustrate, reference figures 4 and 5 below, which showcase the Mandelbrot set at different zoom levels. Note that the higher zoom level reveals a highly similar image to the image given at a zoom level of x1.

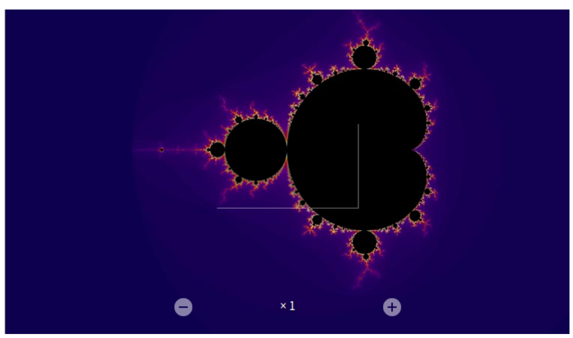

Figure 4: An image of the Mandelbrot set at x1 magnification (Mathigon).

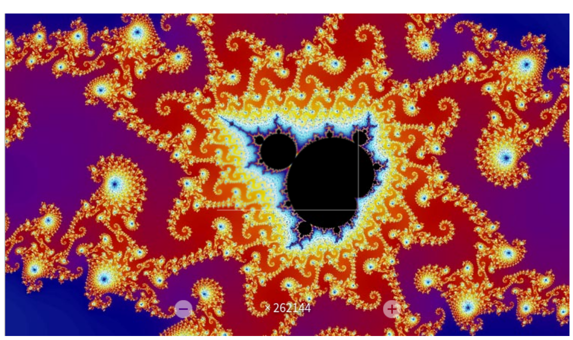

Figure 5: An image of the Mandelbrot set at x262144 magnification (Mathigon).

2.2.4 Infinite Length in a Finite Area

Line-based fractals often have the property of having infinite length in a finite area, that is, they form an infinitely long line which does not take up an infinitely large space. The Koch Curve again is a prime example of this. At each 'level' of the fractal, the length of the line increases, as the sum of the two angled segments' lengths is greater than that of the removed $\frac{1}{3}$ of the original line. But, as we can see, even an arbitrarily large number of iterations to generate an incredibly complex Koch Curve would not extend it far beyond the bounds exhibited in figure 3. This is because each iteration takes up considerably less space than the last, though many more segments are extended, and thus more length added at each level (Wahl 1995).

2.2.5 Space-Filling Curves

It is also possible to create fractal curves which can completely fill a 2-dimensional space, despite being lines themselves. A relatively simple example of this is the Hilbert Curve, pictured below. The curve, though constructed of 1-dimensional line segments, fills the entire 2-dimensional square after an infinite number of iterations.

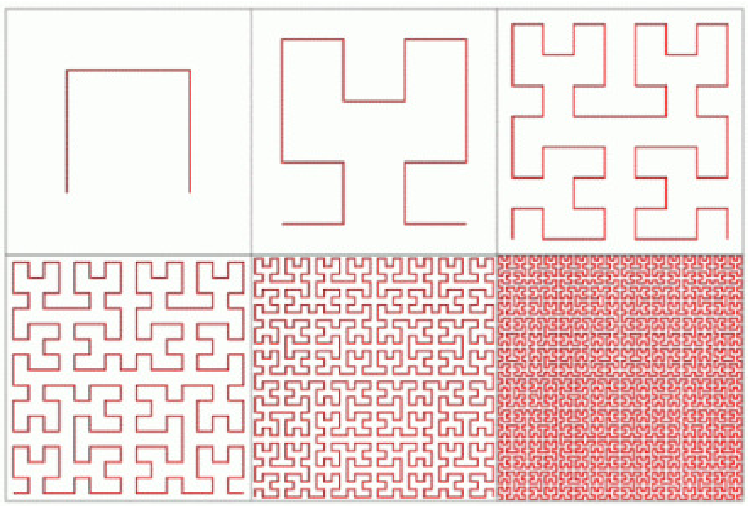

Figure 6: The Hillbert Curve (e-olymp 2019)

3 Applications

Of course, however interesting and unique fractals may be, what matters most is how they may be applied to solve real-world problems. Fortunately, fractals are very useful in many different ways, lending greatly to computer science and advanced simulations. Likely there are applications we have yet to discover.

3.1 Computer Science & Graphics Generation/Image Compression

The principal application of fractals is their use in computer science for image compression. Fractals, by nature, build off themselves in self-similar ways, much like the majority of the natural world. As such, they may be used to model portions of images, which should contain natural fractal structures, therefore reducing the overall amount of data which needs to be stored. For example (these numbers are not exact, just examples), if one wanted to store an image of a mountain range, they could store exact pixel values for the entire image, or they could have a computer generate a rough fractal function to fit the mountains, and then store data to mold the rough approximation into the original image. This generally takes much less processing power, as it reduces the overall number of operations required by the computer, and allows the image to be stored in a smaller size, replacing potentially millions of bits of data with a simple equation (Mathigon).

3.2 Representations of Self-Similar Processes

Due to the self-similar properties of fractals, they may be used to represent various natural and non-natural processes with likewise self-similar properties. For example, most have witnessed the formation of frost crystals across a surface, like those which appear on car windshields on cold winter mornings. An engineer might wish to model the formation these crystals to optimize the performance of a high-altitude drone. To create this model, they would use fractal equations to best fit the expansion of the frost crystals, which follows a fractal pattern. Without a fractal representation, the simulation would be extremely limited in resolution, rather than having the infinite resolution of fractals, and would require a significantly more complicated algorithm. The same concept may be applied to other processes, such as crystal manufacturing, whose formation processes also follow fractal patterns. Modeling of natural objects, like leaves or the structure of human lungs may be modeled using fractals as well, allowing for in-depth computer analysis of them to discover new solutions or applications (Bassingthwaighte, J. B., Glenny, R. W., Robertson, H. T., & Yamashiro, S., 1985).

4 Representation In Artwork and Popular Culture

Of course, such an interesting and beautiful mathematical concept has a sizable following in popular culture. Foremost is the prevalence of the most popular and attractive fractals in the public eye. Most people are at least aware of the Mandelbrot set (See figures 4 and 5), and fewer, but still a significant amount, know of the Julia set (pictured below), another popular fractal. Often these are included in media, and are some of the first examples someone wishing to learn about fractals may come across. Mandelbrot himself is even the subject of popular jokes, such as "The B in Benoît B. Mandelbrot stands for Benoît B. Mandelbrot."

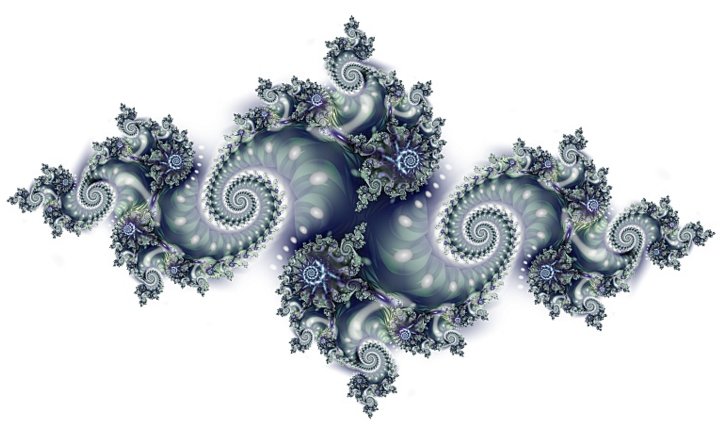

Figure 7: The Julia Set (Design Kompany, October 1 2018)

Also notable is the use of fractals in cinematography, utilizing their appeal to the human eye. Fractals look interesting, so they are often included in movies where appropriate. For example, note the appealing ‘fractalesque’ spiral in the right side of this scene from Marvel’s ‘Doctor Strange’ (2016). The spiral helps grab the viewer’s attention and give a more mysterious look to the scene.

Figure 8: Still image from Marvel’s ‘Doctor Strange’ (2016) (Fawcett, November 21 2016)

5 Conclusion

Fractal geometry holds some of the most strange and incredible properties that mathematics has to offer. They have fractional dimensions instead of the integer dimensions we use regularly, existing somewhere in between our dimensions. This gives them their odd properties, such as filling 2-dimensional spaces with 1-dimensional objects and sometimes containing full copies of themselves within themselves. We can use these properties in a number of applications, generating models of ice crystals or compressing images to make computer storage more efficient. Some may even be colorized and turned into artwork, like the famed Mandelbrot Set. Regardless of application, fractals are a unique class of geometry which have earned their place in the history of mathematics, and can represent our world like no other.

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