13.6 Surface Integrals & Flux

13.6.1 Surface Integrals

We've previously seen double integrals used for surface area.

Definition

If S is a smooth parametric surface with equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

where $(u,v) \in D$, and S is covered just once as (u,v) ranges through D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

where

$$\mathbf{r}_u = x_u\mathbf{i} + y_u\mathbf{j} + z_u\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_v = x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k}$$

On the other hand, if $z = f(x,y)$, where $(x,y) \in D$ and f has continuous partial derivatives, then

$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dA$$

We expand on this in a way that we typically have before. Suppose D is a rectangle in the uv-plane and S is a surface in xyz-space with vector equation $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, where $(u,v) \in D$. Further suppose f is a scalar field on $\mathbb{R}^3$. Then we can subdivide D into mn subrectangles of dimension $\Delta m \times \Delta n$. THe ijth subrectangle of D then corresponds to S_{ij}, the patch on the surface S with corresponding area ΔS_{ij}. The following definite integral then can be defined.

$$\iint_S f(x,y,z) \, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \, \Delta S_{ij}$$

We have previously seen that $\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|$. Knowing this, we can make the following definition.

Definition

Suppose a surface S has vector equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

where $(u,v) \in D$. Define the **surface integral of** f **over the surface** S to be

$$\begin{aligned}\iint_S f(x,y,z)\ dS &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij} \\ &= \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\ dA\end{aligned}$$

This integral represents the accumulation of f over the surface S, analogous to a 2-dimensional version of a line integral. Note that we are integrating over the surface S in $\mathbb{R}^3$, not the planar region in the uv-plane.

Note: $\iint_S 1\ dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\ dA$

Example 1. Compute the surface integral $\iint_S xyz\ dS$, where S is the cone $x = u\cos v$, $y = u\sin v$, $z = u$ with $0 \le u \le 1$ and $0 \le v \le \frac{\pi}{2}$.

Theorem

If a surface S has equation $z = g(x, y)$, where g has continuous partial derivatives, then

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Example 2. Let S be the part of the plane $2x + 2y + z - 4 = 0$ in the first octant. Evaluate $\iint_S xz\, dS$.

Definition

A surface S is called **piecewise-smooth** if it is the finite union of smooth surfaces that intersect only along their boundaries.

Theorem

If a surface $S = S_1 \cup S_2 \cup \cdots \cup S_n$ is piecewise-smooth, then

$$\iint\limits_S f(x,y,z)\ dS = \iint\limits_{S_1} f(x,y,z)\ dS + \iint\limits_{S_2} f(x,y,z)\ dS + \cdots + \iint\limits_{S_n} f(x,y,z)\ dS$$

Example 3. Evaluate $\iint_S z\ dS$, where S is the surface whose sides S_1, S_2, and S_3 are given respectively as the cylinder $x^2 + y^2 = 1$ (the side), the disk $x^2 + y^2 \le 1$ in the xy-plane (the bottom), and the part of the plane $z = 1 + x$ above S_2 (the top).

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13.6.2 Flux

As with any integral, we need a form of an orientation. Two famous examples of nonorientable surfaces are a M obius Strip and a Klein Bottle.

We think about orientation of surfaces this way. Orientable surfaces have two sides, and a nonorientable surface has only one. Here is a more robust definition.

Definition

A surface S is called **oriented** if it is possible to choose a unit normal vector $\mathbf{n}$ at every point (x, y, z) such that $\mathbf{n}$ varies continuously over S. Choosing such an $\mathbf{n}$ provides S with an **orientation**.

Note: For a surface S produced by $z = g(x, y)$, one orientation can be found by

$$\mathbf{n} = \frac{-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + g_x^2 + g_y^2}}$$

Another orientation would be $-\mathbf{n}$.

For a surface S produced by $\mathbf{r}(u, v)$, one orientation can be found by

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

Definition

A surface is **closed** if it includes all of its boundary. A surface is **open** if it includes none of its boundary.

Definition

A closed surface has **positive orientation** if its normal vectors point outward from the surface (same direction as the position vector), and the inward-pointing normal vectors give the negative orientation (opposite direction as the position vector).

To help provide context for a surface integral (or flux or flux integral), we imagine this application.

Suppose S is an oriented surface with unit normal $\mathbf{n}$. Suppose also that a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flows through S. Now, S is not impeding this flow – S is merely present as the fluid flows. Then we can describe the flow (with units of mass per time) per unit of area as $\rho\mathbf{v}$.

Subdivide the surface S into patches S_{ij} as usual. As the number of patches increases, S_{ij} resembles a planar region, and so the *mass of fluid per unit time passing through S_{ij} in the direction of* $\mathbf{n}$ is

$$(\rho\mathbf{v} \cdot \mathbf{n})\ A(S_{ij}) = (\rho(x_{ij}, y_{ij}, z_{ij})\mathbf{v}(x_{ij}, y_{ij}, z_{ij}) \cdot \mathbf{n}(x_{ij}, y_{ij}, z_{ij}))\ A(S_{ij})$$

where (x_{ij}, y_{ij}, z_{ij}) is a point on S_{ij}. As with all integrals, we define some new integral as the limit of the sum of these approximations. That is, this integral represents the limit of the sum of the mass of fluid per unit time passing through S in the direction of $\mathbf{n}$; that is, this integral represents the rate of flow through S.

$$\iint\limits_S \rho\mathbf{v} \cdot \mathbf{n}\ dS = \iint\limits_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\ dS$$

Now, if $\rho\mathbf{v}$ is a vector field on $\mathbb{R}^3$, and we can abstract this to any vector field $\mathbf{F}$ on $\mathbb{R}^3$.

Definition

Suppose S is an oriented surface with unit normal $\mathbf{n}$. If $\mathbf{F}$ is a continuous vector field defined on S, then the **flux of F across** S is given by

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_S \mathbf{F} \cdot \mathbf{n}\ dS$$

This integral is also called the **surface integral of F over** S. We interpret this as *the flux of* $\mathbf{F}$ *over S is the flux of the normal component of* $\mathbf{F}$ *over S.*

Definition

Suppose S is an oriented surface produced by $\mathbf{r}(u, v)$ with parameter domain D. If $\mathbf{F}$ is a continuous vector field defined on S, then

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA$$

Example 4. Find the flux of $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the unit sphere. *Note that if* $\mathbf{F}$ *is the velocity field describing the flow of a fluid with density 1, then the conclusion represents the total rate of flow through the unit sphere in mass per time.*

Suppose $z = g(x, y)$ describes a surface in $\mathbb{R}^3$. Then

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = \langle P, Q, R\rangle \cdot \langle -g_x, -g_y, 1\rangle$$

Definition

Suppose S is an oriented surface produced by $z = g(x, y)$, where g has domain D. Then

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k})$$

and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (-Pg_x - Qg_y + R)\ dA$$

Example 5. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = \langle y, x, z \rangle$ and S is the boundary of the solid enclosed by $z = 1 - x^2 - y^2$ and the xy-plane.