12.6 Parametric Surfaces

12.6.1 Parametric Surface – Preliminary Work for Surface Area

Goal: Find the surface area of a parametric surface S.

Recall that a parametric surface S is defined by a vector function of two variables, say

$$\begin{aligned}\mathbf{r}(u,v) &= x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \\ &= \langle x(u,v), y(u,v), z(u,v)\rangle\end{aligned}$$

where $(u,v) \in D$, a region of the uv-plane.

Let's subdivide S into "patches" and approximate the surface area of each patch. We will approximate the surface area of each patch with a small parallelogram tangent to S.

Let P_0 be a point on S with position vector $\mathbf{r}(u_0, v_0)$. Keeping u constant with $u = u_0$, then $\mathbf{r}(u_0, v)$ depends only on a single parameter, producing a grid curve C_1 on S. We can find a tangent vector along C_1 at P_0 by finding $\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$. For shorthand, we will abbreviate $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$. That is,

$$\begin{aligned}\mathbf{r}_v &= \frac{\partial x}{\partial v}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\mathbf{k} \\ &= \langle x_v(u_0,v_0), y_v(u_0,v_0), z_v(u_0,v_0)\rangle\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{r}_u &= \frac{\partial x}{\partial u}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0,v_0)\mathbf{k} \\ &= \langle x_u(u_0,v_0), y_u(u_0,v_0), z_u(u_0,v_0)\rangle\end{aligned}$$

In order to adopt a bit more convention of language, we introduce this definition.

Definition

Let S be the parametric surface determined by $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$. Let the **normal vector** to S at (u_0, v_0) be $\mathbf{r}_u \times \mathbf{r}_v$. If the normal vector to S at (u_0, v_0) is not $\mathbf{0}$, we say that S is **smooth** at (u_0, v_0).

Theorem

If a parametric surface S is smooth at a point P, then there exists a tangent plane to S at P, and it can be found using the normal vector.

12.6.2 Parametric Surface – Building a Double Integral for Surface Area

Consider a surface S defined over a rectangle D. Subdivide D into mn subrectangles R_{ij} of width Δu and length Δv, and respectively subdivide S into mn "patches" S_{ij}. In this way, S_{ij} corresponds to R_{ij}. Choosing (u_{ij}^*, v_{ij}^*) in each R_{ij} to be lower-left corners, computations will be a bit simpler. Then for each (u_{ij}^*, v_{ij}^*), we have a corresponding position vector $\mathbf{r}(u_{ij}^*, v_{ij}^*)$ drawn from the origin to the lower-left corner of each patch, P_{ij}, where

$$P_{ij} = (x(u_{ij}^*, v_{ij}^*), y(u_{ij}^*, v_{ij}^*), z(u_{ij}^*, v_{ij}^*))$$

Define $\mathbf{r}_u^* = \mathbf{r}_u(u_{ij}^*, v_{ij}^*)$ and $\mathbf{r}_v^* = \mathbf{r}_v(u_{ij}^*, v_{ij}^*)$. Then $\mathbf{r}_u^*$ is a tangent vector to S_{ij} at P_{ij} in the direction of u, and $\mathbf{r}_v^*$ is tangent to S_{ij} at P_{ij} in the direction of v. These two tangent vectors determine a parallelogram Π_{ij}

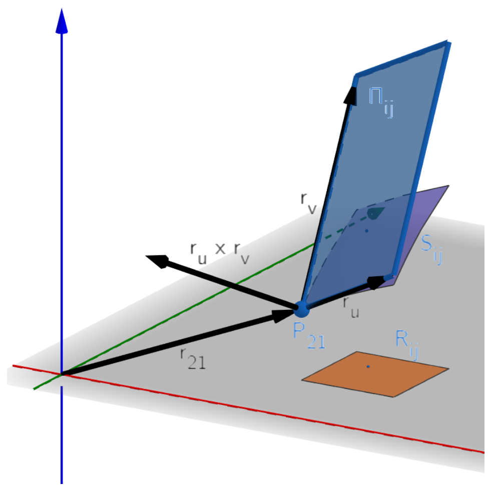

Figure 1: https://www.geogebra.org/3d/nhvnn4rv

Now, the area of Π_{ij} can be found, and it approximates the surface area of S_{ij}. In particular,

$$\begin{aligned}\text{Area}(S) &= \sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij} \\ &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} \Pi_{ij}\end{aligned}$$

From MTH 253, $|\mathbf{a} \times \mathbf{b}|$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. In particular, the sides of the parallelograms Π_{ij} are $\Delta u \mathbf{r}_u^*$ and $\Delta v \mathbf{r}_v^*$. It follows that

$$\begin{aligned}\text{Area}(\Pi_{ij}) &= |(Deltau\mathbf{r}_u^*) \times (\Delta v \mathbf{r}_v^*)| \\ &= |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \; \Delta v\end{aligned}$$

Moreover,

$$\begin{aligned}\text{Area}(S) &= \sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij} \\ &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} \Pi_{ij} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \; \Delta v\end{aligned}$$

Definition

If S is a smooth parametric surface determined by

$$\begin{aligned}\mathbf{r}(u,v) &= x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \\ &= \langle x(u,v), y(u,v), z(u,v)\rangle\end{aligned}$$

where $(u,v) \in D$, and S is covered just once as (u,v) ranges throughout the parameter domain D, then the **surface area** of S is

$$\begin{aligned}\text{Area}(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \, \Delta v \\ &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA\end{aligned}$$

where

$$\begin{aligned}\mathbf{r}_u &= \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \\ &= \langle x_u, y_u, z_u\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{r}_v &= \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} \\ &= \langle x_v, y_v, z_v\rangle\end{aligned}$$

Example 1. Find the surface area of the portion of the cone $x^2 + y^2 = z^2$ above the disk $x^2 + y^2 = 4$.

12.6.3 Level Surface – Double Integral for Surface Area

Let S be the surface whose equation is $z = f(x, y)$, where f has continuous partial derivatives. For simplicity of computation, we will assume $f(x, y) \geq 0$ for all $(x, y) \in D$, a rectangle that S is defined over.

If we follow the same steps as before, then we arrive at a very similar picture that will guide us to our formula. We will subdivide D into mn rectangles R_{ij}, and we will respectively subdivide S into mn patches S_{ij} with lower-left corners being $P_{ij}(x_i, y_j, f(x_i, y_j))$.

If we compute tangent vectors to S_{ij} at P_{ij} in the x and y directions, we get

$$a = \Delta x \ \mathbf{i} + f_x(x_i, y_j)\Delta x \ \mathbf{k}$$
$$b = \Delta y \ \mathbf{i} + f_y(x_i, y_j)\Delta y \ \mathbf{k}$$

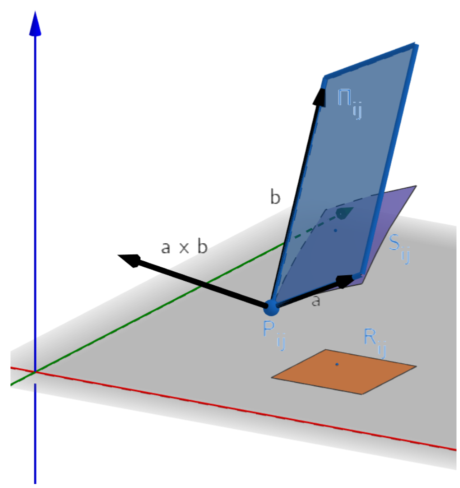

Figure 2: https://www.geogebra.org/3d/bdpeprce

Again, we have that the area of Π_{ij} can be found, and it approximates the surface area of S_{ij}. In particular,

$$\begin{aligned}\text{Area}(S) &= \sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij} \\ &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} \Pi_{ij}\end{aligned}$$

The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$, so it follows that $\text{Area}(\Pi_{ij}) = |\mathbf{a}\times\mathbf{b}|$.

In order to simplify,

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\Delta y \end{vmatrix} \\ &= (-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k})\Delta A \\ |\mathbf{a}\times\mathbf{b}| &= \sqrt{((f_x(x_i, y_j))^2 + (f_y(x_i, y_j))^2 + 1} \ \Delta A\end{aligned}$$

Definition

If S is the surface whose equation is $z = f(x,y)$, where f has continuous partial derivatives and domain D, then

$$\begin{aligned}\text{Area}(S) &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\sqrt{((f_x(x_i,y_j))^2 + (f_y(x_i,y_j))^2 + 1}\,\Delta A \\ &= \iint_D \sqrt{((f_x(x,y))^2 + (f_y(x,y))^2 + 1}\,dA \\ &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA\end{aligned}$$

Example 2. Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Example 3. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$.