# 254 Prerequisite Topics

## 254.1 Vector Functions

Given a vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$,

$$\begin{aligned} \mathbf{r}(t) &= \langle f(t), g(t), h(t) \rangle \\ \mathbf{r}'(t) &= \langle f'(t), g'(t), h'(t) \rangle \\ \int_a^b \mathbf{r}(t)\ dt &= \left\langle \int_a^b f(t)\ dt, \int_a^b g(t)\ dt, \int_a^b h(t)\ dt \right\rangle \\ \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \end{aligned}$$

$$\begin{aligned} \kappa &= |\frac{d\mathbf{T}}{ds}| \\ &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \\ &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \\ \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \end{aligned}$$

**Example 1.** Find the curvature of $\mathbf{r}(t) = \langle t^2, \ln t, t \ln t \rangle$ at the point $(1, 0, 0)$. What does this number represent?

## 254.2 Differentiable Multivariable Functions

Given a differentiable multivariable function $f(x, y, z)$,

$$f_x(x,y,z) = f_x = \frac{\partial f}{\partial x}$$
$$f_y(x,y,z) = f_y = \frac{\partial f}{\partial y}$$
$$f_z(x,y,z) = f_z = \frac{\partial f}{\partial z}$$
$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$f_{xx}(x,y,z) = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x$$
$$f_{xy}(x,y,z) = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y$$

A **level surface** of $f$ is a surface whose equation is $f(x, y, z) = k$, where $k \in \mathbb{R}$.

Given a multivariable function $g(x, y)$, the graph of $g$ is a surface. Call this surface $S$. Suppose $z = g(x, y)$. Then

The **tangent plane** to $S$ at a point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

**Example 2.** Find an equation for the tangent plane to $\cos^2 x + \sin^2 y + e^z \ln z = 1$ at the point $(1, 1, 1)$.

## 254.3 Double Integrals

Suppose $f$ is integrable on its domain.

- If $R$ is the Cartesian rectangle $[a,b] \times [c \times d] = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$, then
$$\iint\limits_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dy\ dx$$
- If $R$ is the polar rectangle $[a,b] \times [\alpha,\beta] = \{(r,\theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$, then
$$\iint\limits_R f(x,y)\ dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta) r\ dr\ d\theta$$
- **Fubini's Theorem:** If $f$ is continuous on the rectangle $R = [a,b] \times [c \times d]$, then
$$\iint\limits_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy$$

**Example 3.** Find the volume of the solid that lies above the quarter-annulus depicted below and beneath the plane $z = x + y$.