13.3 Fundamental Theorem of Line Integrals

13.3.1 The Fundamental Theorem for Line Integrals

The Fundamental Theorem for Line Integrals

Let C be a smooth curve whose vector function is $\mathbf{r}(t)$ with $t \in [a, b]$. Let f be a differentiable function whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f[\mathbf{r}(b)] - f[\mathbf{r}(a)]$$

That is, to evaluate a line integral over a conservative vector field, find a potential function, evaluate it at the endpoints, and subtract.

Thus, the line integral of a conservative vector field depends only on the initial and terminal points of a curve.

Note:

- In $\mathbb{R}$, $\int_C \nabla f \cdot d\mathbf{r} = f(x_2) - f(x_1)$.
- In $\mathbb{R}^2$, $\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$.
- In $\mathbb{R}^3$, $\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$.

Proof:

13.3.2 Path Independence

Definition

If C is a piecewise-smooth curve with initial point A and terminal point B, then we call C a **path from** A **to** B.

Example 1. Let C be a path from $(1, 0)$ to $(-1, 0)$ along the unit circle. Find

$$\int_C \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle \cdot d\mathbf{r}$$

Definition

If $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path-independent** (or **independent of path**) if

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

for any two paths C_1 and C_2 in D that have the same initial and terminal points.

Example 2. Let C be a path from $(1, 0)$ to $(-1, 0)$ along the parabola $y = 1 - x^2$. Find

$$\int_C \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle \cdot d\mathbf{r}$$

Theorem

Let ∇f be continuous. If C_1, C_2 are two paths from A to B, then

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

Definition

A path is called **closed** if its terminal point coincides with its initial point.

Theorem

The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent in D iff $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

All of this is to say that line integrals in a conservative vector field are significantly nicer to compute that in a general vector field. So how can we identify when a vector field is conservative?

Definition

A set D in $\mathbb{R}^3$ is **open** if for every point $P \in D$, there is a disk with center P that lies entirely in D.

Definition

A set D in $\mathbb{R}^3$ is **connected** if for any two points in D there is a path in D that connects them.

Theorem

Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path-independent in D, then $\mathbf{F}$ is a conservative vector field on D. That is, there exists a potential function for $\mathbf{F}$. That is, there exists a function f such that $\nabla f = \mathbf{F}$.

13.3.3 Simply-Connected Regions

Theorem

If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on D, then for all $(x, y) \in D$,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Definition

A curve that does not intersect itself between its endpoints is called a **simple curve**.

Definition

Let D be a planar region. We say that D is a **simply-connected region** if every simple closed curve in D encloses only points in D.

Theorem

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. If P, Q have continuous first-order partial derivatives and for all $(x, y) \in D$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

then $\mathbf{F}$ is conservative.

Note: We use the above theorem to determine whether a field is conservative or not. This is one of the big goals we wished to achieve.

Note: Potential Function and Conservative Vector Field are analogous to Potential Energy and Conservation of Energy in physics.

Example 3. Determine whether the vector field is conservative or not.

$$\mathbf{F}(x, y) = (2x + 3x^4y^5)\mathbf{i} + (-6y + 3x^5y^4)\mathbf{j}$$

13.3.4 Partial Integration

We've seen partial differentiation, and now we are looking for potential functions of a conservative vector field, so we introduce the idea of partial integration.

Example 4. Suppose $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$.

a. Determine if $\mathbf{F}$ is conservative or not.

b. If $\mathbf{F}$ is conservative, find a potential function f for $\mathbf{F}$.

c. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by

$$\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t \rangle \quad , \quad t \in [0, \pi]$$

Example 5. Find a potential function for $\mathbf{F}(x, y, z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$.