Tessellations: The Art of Math

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Abstract

The mathematics of tessellations are used in many different careers, including many that are not typically associated with having to have a strong background in mathematics. Some of the varying careers where tessellations are often used in are: architecture, tile laying, fashion designing, and carpet making; careers that are strong in designing.

Before the mathematics of tessellations were used in designing, they were first observed in nature: in hexagonal-patterned honeycombs, in flowers that bloom in tessellated patterns, and in rocks that fracture or break into a pattern of rectangular blocks, just to name a few.

Tessellations create a bridge between science, art, and nature in the form of mathematics, and their repeating patterns have captured our fascination for millennia.

The study of mathematics in school is often viewed with disdain by students who believe that they will never use the mathematics that they have learned for their future careers, and the only math that they will need to do is one that they can plug quickly into a calculator. Tessellations are a form of mathematics that defy that notion, and appear in many vastly different careers; from a tiler laying tile in a new building, to a fashion designer creating a pattern for their new line. While some careers may use tessellations more readily than others, it is a mathematical art form that is used by designers–whether their designs are more artistically based, or mathematically based–and it is a bridge that connects these otherwise disparate careers.

There are many types of tessellations, and they all can be created in many different ways, but in a general sense, a tessellation is a flat canvas of some type that is covered by a pattern of a repeated shape, or shapes, without any overlapping of the pattern or any of the canvas being left blank, to create a perfect pattern [7]. While there are different types of tessellations, including ones that look much more complicated than others, they all follow the same basic rule: to make a tessellating pattern, "the sum of the [shapes being used for the pattern's] interior angles that meet at a common point must equal 360 degrees" [1]. Some forms of tessellations require more steps to create them, like the pieces of artwork that are tessellating patterns of shapes other than regular polygons, but all tessellations revolve around the same idea, with the more complicated designs incorporating a more in-depth approach with the basic rule.

There are quite a few different types of tessellating patterns, with some being more commonly, or naturally, occurring than others; some that have to be manipulated to create their pattern; and some commonly occurring ones that can be manipulated to make other commonly occurring tessellations. All types of tessellations fall into different overarching categories with other tessellations, but each one still manages to be its own type of tessellating pattern that has its own rules that the other tessellations in its category do not follow. Before breaking down some of the categories into their more specific types of tessellations, there are three notable categories of tessellations: regular tessellations, semi-regular tessellations, and non-regular tessellations, with some cross-over between the categories and their tessellating patterns. [5][7].

The most commonly occurring tessellations are regular tessellations. A regular tessellation is a tessellating pattern that consists of only one type of regular polygon [2]. A regular polygon is a polygon that has only equal-length sides and equivalent angles, like a square: four sides of equal length, and all of its angles are 90 degrees. With that being said, there are only a few regular polygons that can be used to create a regular tessellation, as there are very few regular polygons that have angles that 360 is divisible by.

The first step to determining whether or not a regular polygon will work for a regular tessellation is to figure out the degree of one of its angles. As it is unlikely to know that off the top of one's head, there is an equation that we can use to determine the degree of a regular polygon's angle [1]. *Note: The following equation can only be used to determine the angles of a regular polygon.*

Let n represent the number of sides that the regular polygon has, and let a represent the degree of the angle.

$$a = \frac{180\,(n-2)}{n} \tag{1}$$

Once we have the angle of our chosen regular polygon, the next step is to see if 360 is divisible by our angle. As it is, there are only three regular polygons that have angles that are factors of 360: an equilateral triangle with 60 degree angles, a square with 90 degree angles, and a regular hexagon with 120 degree angles. These three regular polygons are the only shapes that can be used to create a regular tessellation [1].

To create each of these shapes' tessellating patterns, and prove that they work, we can use transformations in $\mathbb{R}^2$. While there are many types of transformations, there are only a few that, when used correctly, will create the beginning of a true regular tessellation on their own. Some of the most notable transformations that will create a regular tessellation are two linear transformations: the rotation transformation and the reflection transformation. As both transformations will work, we are able to utilize either. In my personal opinion, I believe that the rotation transformation is the most beneficial method to use, as a reflection transformation requires finding the slope of some side lengths of the chosen polygon to use in the reflection transformation, with the standard reflection transformation matrix changing with each new mapping of the polygon, all in addition to finding the exact coordinates of each point of the polygon. The rotation method uses the same standard rotation transformation matrix throughout its mapping process, and it does not require finding the slope of a given side length of the polygon. It also is easier to use when wanting to use a translation transformation for other tessellating patterns. For that reason, I will only be using rotation transformations.

A rotation transformation in $\mathbb{R}^2$ is a "...transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle" [3]. To use a rotation transformation to create the beginning of regular tessellation, the first step is to create the vertices of the regular polygon by using vectors, while making sure that one of the vertices is on the origin, or, in other words, its vector is the zero vector, and thus it will be the point of rotation for the polygon. After all the vertices have been determined, identify the degree of an inner angle of the chosen regular polygon, being 60, 90, and 120 degrees for the equilateral triangle, square, and regular hexagon, respectively. The degree of the angle is equivalent to the angle of rotation, φ, that each iteration of the polygon will rotate about the origin to create a regular tessellating pattern. Once the angle of rotation and all of the vectors of the polygon have been determined, let each vector be transformed by the standard rotation transformation matrix A, which is:

$$A = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

[3]. Once each vector of the polygon has been mapped to a new vector, it can be seen that by looking at the new vectors as position vectors, they can be used to create the vertices of a regular polygon congruent to the original, and that both polygons share a vertex point, or position vector, at the origin, along with a vector that leads to another shared vertex point. To continue the pattern, follow the same steps as above, except, instead of using the vectors of the original regular polygon, use the vectors from the most recently mapped polygon, and keep going until the most recently mapped polygon shares two vertices and a vector with the original polygon, not including the polygon mapped from the first transformation.

Example:

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates a square with vertices $\mathbf{v_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v_4} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and each iteration of its transformation after that about the origin through an angle of $90°$, or $\frac{\pi}{2}$ radians, until the most recently mapped square shares two vertices and a vector with the original square, not including the square mapped from the first transformation.

Solution:

Through prior knowledge, we know that $\mathbf{0}$ will always be $\mathbf{0}$, as $\mathbf{0}$ multiplied by any scalar c, or any matrix A, will result in $\mathbf{0}$. Thus, each transformation of each iteration of $\mathbf{v_1}$ will always be $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so we do not need to perform any transformations on $\mathbf{v_1}$; only on each iteration of $\mathbf{v_2}$, $\mathbf{v_3}$, and $\mathbf{v_4}$, until we get a tessellation.

$$T(\mathbf{v_2}) = A\mathbf{v_2} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{v_2'}$$

$$T(\mathbf{v_3}) = A\mathbf{v_3} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \mathbf{v_3'}$$

$$T(\mathbf{v_4}) = A\mathbf{v_4} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \mathbf{v_4'}$$

The transformation maps the new vectors $\mathbf{v_1'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v_2'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{v_3'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{v_4'} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, and it is a square. Now perform the same transformation on each iteration, until we have a tessellation.

$$T(\mathbf{v_2'}) = A\mathbf{v_2'} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \mathbf{v_2''}$$

$$T(\mathbf{v_3'}) = A\mathbf{v_3'} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \mathbf{v_3''}$$

$$T(\mathbf{v_4'}) = A\mathbf{v_4'} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathbf{v_4''}$$

The new vectors are $\mathbf{v_1''} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v_2''} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{v_3''} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\mathbf{v_4''} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, and it is a square. The new square created only shares $\mathbf{0}$ with the original, so we cannot stop here.

$$T(\mathbf{v_2''}) = A\mathbf{v_2''} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathbf{v_2'''}$$

$$T(\mathbf{v_3''}) = A\mathbf{v_3''} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbf{v_3'''}$$

$$T(\mathbf{v_4''}) = A\mathbf{v_4''} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{v_4'''}$$

The new vectors are $\mathbf{v_1'''} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{v_2'''} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\mathbf{v_3'''} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v_4'''} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and it is a square. Now we can see that $\mathbf{v_2} = \mathbf{v_4'''} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v_1} = \mathbf{v_1'''} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and they share the same side along the x-axis, meaning that the transformation has made it back to the original square, so we do not have to perform anymore rotation transformations, and the square does indeed create a regular tessellation. The graph down below shows each mapping of the polygon.

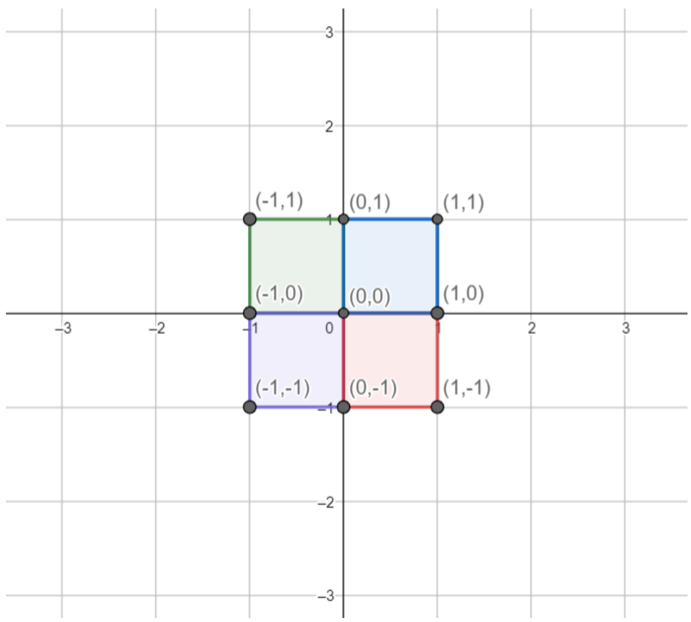

Figure 1: Rotation Transformations

□

The above equations must be adjusted for an equilateral triangle tessellation and a regular hexagonal tessellation, as both polygons do not have the same amount of vertices as a square, nor will rotating the polygons about the origin by 90°, or $\frac{\pi}{2}$ radians, work. Once the appropriate adjustments are made, though, we can see that using a rotation transformation on both of the regular polygons work, and that they make regular tessellations.

While there may only be three regular polygons that work for regular tessellations, there are eight more tessellating patterns that can be created by using regular polygons, and these tessellations are known as semi-regular tessellations [2].

Semi-regular tessellations are tessellations that are composed of more than one regular polygon, but still follow the same basic rules that all tessellations must follow [2]. Their repeated patterns are based around the vertices of the polygons, as the pattern at each vertex must be the same for the tessellation to be a semi-regular tessellation, and that is where each of the eight semi-regular tessellations get their names. The eight semi-regular tessellations are all named after the polygons that surround each vertex, or, more specifically, the number of sides each polygon has. Starting with the smallest numbered sides, and then going around the vertex from that point in the direction of whichever provides the next smallest numbered sides of the polygon. If there are multiple polygons that all share the least amount of sides surrounding the vertex, the one that is listed first is based on the neighbors of these polygons; more specifically, whichever neighbor has the least amount of sides, and will consequentially be the second part of the semi-regular tessellation's name. The eight semi-regular tessellations are: "3.3.3.3.6", "3.3.3.4.4", "3.3.4.3.4", "3.4.6.4", "3.6.3.6", "3.12.12", "4.6.12", and "4.8.8" [7].

Just as with regular tessellations, to create a semi-regular tessellation and prove that they work, we can use rotation and reflection transformations in $\mathbb{R}^2$, with some tessellations being more complicated to map than others. Since semi-regular tessellations are composed of at least two types of regular polygons, the process of mapping the transformations to make the tessellating patterns has to be approached with different set-up requirements, which will lead to different end requirements, and can be done, more or less, in a variety of ways. While many of the different ways of mapping the transformations may not technically be mathematically true to a semi-regular tessellation, they can still visually produce semi-regular tessellations.

One of the biggest differences from set-up is that not every polygon in a semi-regular tessellation can have one of their vertices located at the origin, and some of the tessellations cannot have any vertices on the origin, depending on the method chosen. As there are multiple polygons in each tessellation, the transformations have to take each polygon in the tessellation into consideration before they can be mapped correctly, so that means that angle of rotation will not be as easily deduced as the angle of rotation for the regular tessellations rotation transformations. For the end result, since not every polygon will have a vertex located at the origin, the two shared vertices of one of the original polygons and its last transformation might not include the origin, so checking the vertices to confirm that it is indeed a semi-regular tessellation has to be more thorough than with the regular tessellations.

The method for transformation that is probably the most true mathematically to semi-regular tessellations is by having one of the polygons, preferably the one that occurs the least amount of times in the pattern, centered around the origin, sharing two of its vertices with the other polygon in the pattern, and only using the linear transformations on the polygon not centered around the origin. If there are more than two polygons in the pattern, keeping the majority of the polygons stationary and around the origin, while only transforming one of the polygons, is still probably the most accurate representation, mathematically, for those tessellations. For a method that still works visually, but is not as true to what a semi-regular tessellation is, is by having one of the regular polygons fully formed by its position vectors, then having only a part of the remaining regular polygons represented through position vectors, that will then go through its own linear transformations, maybe following the same steps as the other polygon that was transformed, or maybe even using a completely different linear transformation, to form the regular polygon that it is a part of. To demonstrate these two methods, let's look at the graphs of the "4.8.8" semi-regular tessellation constructed by using both approaches. As the math behind the rotation transformation method has already been demonstrated with regular tessellations, the math will be omitted this time.

First, let's look at the graph with the square centered around the origin, and only the octagon being mapped by the rotation transformation in each iteration, around the origin.

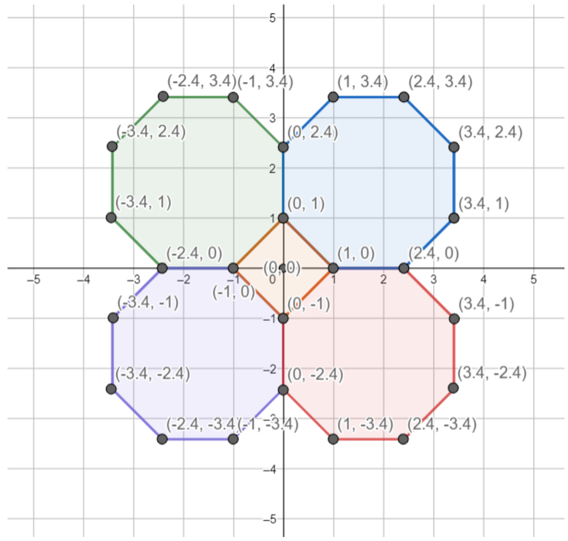

Figure 2: First Method

As can be seen, the octagon rotates nicely around the square, and it creates the beginning of the “4.8.8” semi-regular tessellation, with only regular polygons being shown on the graph.

Now, let’s look at the graph where the octagon and the triangle formed between the x-axis, the y-axis, and the side of the octagon, are both mapped by rotation transformations about the origin.

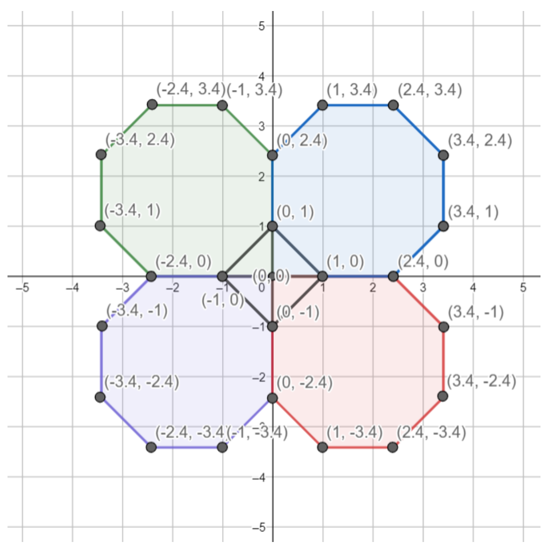

Figure 3: Second Method

The second method’s graph looks similar to the graph from the first method, but by looking at the middle closely, we can see that the “square” is not actually a square in this case, but rather four right triangles that look like a square, meaning that this is technically not a semi-regular transformation. It can still be used to show that some regular polygons create tessellations together, as long as it is understood that it is purely for visual comprehension.

Both regular tessellations and semi-regular tessellations are found in many places, like a kitchen from the 1950s might have the “4.8.8” semi-regular tessellation as its tiling, or someone walking down the street could be wearing a gingham shirt, which is just the square tessellation in different colors. The square tessellation was even the first tiling method to be used thousands of years ago, which is why tessellations are named as such, as the Greek word “tesseres” means “four” [2]. These tessellations have stood the test of time, and they have never truly gone out of style. If anything, humanity has done their utmost to discover all of the tessellations that could exist, and thus, non-regular tessellations have also made their mark on the world.

Non-regular tessellations is the overarching name for any type of tessellation that is neither a regular tessellation, nor a semi-regular tessellation, though they could be modified forms of those tessellations. The four main types of non-regular tessellations are the monohedral tessellations, duals, modified monohedral tessellations, and aperiodic tessellations [2]. Unlike the regular and semi-regular tessellations, the non-regular tessellations have infinitely many tessellation patterns, a few times over.

Monohedral tessellations are similar to regular tessellations in that their patterns may consist of only one polygon, and they follow the same basic rule that all tessellations must follow, but they are dissimilar because the polygon may not be a regular polygon, as that would create a regular tessellation instead. The only polygons that work for monohedral tessellations are triangles, quadrilaterals, pentagons, and hexagons, as the only way a polygon with seven sides or more could be part of a monohedral tessellation is if it were a concave polygon, and even then, not many would work. As of now, there are only 15 known types of pentagons that will tessellate and three known types of hexagons that will tessellate, but mathematicians are looking for more. Unlike the pentagons and hexagons, the triangles and quadrilaterals have infinitely many tessellations, as every non-regular triangle polygon and non-regular quadrilateral polygon produces at least one type of monohedral tessellating pattern, with many of the three-sided and four-sided polygons producing many types of monohedral tessellations [2]. Many of these monohedral tessellations can be created by the same linear transformations as regular tessellations, but there are also many that can only be created when two types of transformations are used in succession, which are better known as affine transformations [9].

An affine transformation equation is similar to a linear transformation equation, as all linear transformations are affine, but not all affine transformations are linear. Whereas the normal linear transformation equation is $T(\mathbf{x}) = A\mathbf{x}$, an affine transformation equation is $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with $\mathbf{b}$ being a translation vector that moves the chosen vector to a new position [9]. Technically, all linear transformations have the transformation equation $T(\mathbf{x}) = A\mathbf{x}+\mathbf{b}$, but the translation vector is $\mathbf{b} = \mathbf{0}$, as for it to be a linear transformation, it must stay on the origin. Translation transformations are also used with the other tessellation types to create the full tessellating pattern, or to create the base shape of a modified monohedral tessellation.

Although every tessellation is unique, as in the way that triangles cannot be used to truly mathematically create the square tessellations, nor the other way around, all of the regular tessellation, semi-regular tessellations, and many of the monohedral tessellations can be redrawn into another type of tessellation known as their duals. To create a tessellation’s dual, draw a dot in the middle of every polygon in the tessellation. Connect these dots by drawing lines from each polygon’s dot to the dot of every other polygon it is touching, and then “erase” the lines from the original tessellation, and a different tessellation will visible. The regular tessellations’ duals are also regular tessellations, as the dual of the square tessellation is another square tessellation, but with a different starting point, and the equilateral triangle tessellation and regular hexagonal tessellation are duals of one another [2]. While duals are not technically a type of tessellation, as all duals are either regular tessellations, semi-regular tessellations, or monohedral tessellations, they still capture the fascination of many mathematicians.

Regular tessellations, semi-regular tessellations, and monohedral tessellations are found all throughout history, whether in the tiling of ancient architecture, or on the picnic blanket of a modern day family out enjoying the sunshine. They have been around for even longer than the first square tiling, since they occur in nature. Modified monohedral tessellations have been around for a relatively long time, as well, but their popularity has come and gone throughout the centuries, with their current popularity being at an all time high, as modified monohedrals tessellations are strongly connected to artwork and the fashion industry.

A modified monohedral tessellation is a tessellating design of one shape, but the shape has been altered from a polygon. In other words, it is a polygon that had translation transformations performed on it to "cut out" sections from the polygon, and move it to a different part of the polygon. This creates a puzzle-piece-like shape that fits into itself, either just through translations transformations, or an additional rotation transformation, which makes affine transformations used here, also [4]. Since these are *modified* monohedral tessellations, they will automatically follow the 360 degree rule, as long as the puzzle pieces are put in the correct places. One of the most famous modified monohedral tessellations is the pattern known as houndstooth, with its oldest known usage on the Gerum Cloak, recently found in Sweden, and it is dated to be from some time between 100 B.C. and 360 B.C. [6]. Houndstooth is used by many designers when they are designing clothing, whether they are designing clothes for the clothing store known as Forever 21, or they are designing for Chanel; houndstooth is a perfect example of a modified monohedral tessellation that is timeless.

While the modified monohedral tessellations like houndstooth are fairly simple in design, the famous twentieth-century Dutch artist, Maurits Cornelis Escher (1898-1972), better known as M.C. Escher, takes modified monohedral tessellations to a whole new level in his art. In his pieces of art, M.C. Escher takes either a regular tessellating pattern or a monohedral tessellating pattern, and he modifies the polygon into an intricate design that will tessellate with itself. One very notable tessellation artwork of his is *Study of Regular Division of the Plane with Reptiles* (1939), which is a modified version of the regular hexagonal tessellation that had gone through many transformations to get it to work. M.C. Escher even created hyperbolic tessellating patterns with modified tessellations, such as his wood engraving *Circle Limit III* (1959), one of a series of four, but he used math to do it [8]. While M.C. Escher certainly had to math knowledge to create the amazing designs, he drew inspiration from the aperiodic tessellations found in medieval Islamic architecture [2].

Aperiodic tessellations are very different from the other tessellating patterns, as the tessellations are not a typical repeated tessellation, and there can be a wide variety of polygons within one tessellation. Aperiodic tessellations still follow the basic rule of 360 degrees that all tessellations must follow, but that is the only thing they have in common with the other tessellations, aside from being made up of polygons. Some of the most famous aperiodic tessellations are found in Islamic architecture and artwork, with many of the patterns predating their discovery in the West by at least 500 years. There are many aperiodic tessellations that still have yet to be discovered to this day. Just in 1936, the first spiral aperiodic tessellation was discovered, and it is unknown how many more there are, or even if all types have been discovered, or not. Five-folded aperiodic tessellations have even been studied for their geometry, and in turn the math started to be applied to crystallography, and it even brought forth the rise in study of quasicrystals in the 1980s [2]. Tessellations as a whole have brought about the further study of math that can be applied to different fields, and it all started in nature.

While humans make a concerted effort to create these mesmerizing patterns, tessellations already occur in nature without any thought of mathematics. One of the most iconic naturally occurring tessellations is the hexagonal-patterned honeycomb created by bees. Bees do not have any education in mathematics, yet they are able to create these perfect mathematical designs. This natural ability to create these designs suggests that there is more to tessellations than just the numbers written down on paper, and perhaps that is why they have held our attention for millennia, and why we aspire to create the mathematical works of art.

References

[1] Guillermo Bautista, *Tessellation: The mathematics of tiling*, 2010. May 15, 2019.

[2] Robert Coolman, *Tessellation: The geometry of tiles, honeycombs and m.c. escher*, 2015. May 15, 2019.

[3] Judi J. McDonald David C. Lay Steven R. Lay, *Linear algebra and its applications, fifth edition*, Pearson, 2016.

[4] Andrew Harris, *The mathematics of tessellations*, 2000. May 15, 2019.

[5] N.A., *What is a tessellation?*, 2018. May 28, 2019.

[6] ———, *Sveriges äldsta bevarade klädesplagg*, N.A. June 10, 2019.

[7] Rod Pierce, *Tessellation*, 2018. May 15, 2019.

[8] Doris Schattschneider, *The mathematical side of m.c. escher*, Notices of the AMS **57** (2010), no. 6, 712–713.

[9] Eric W. Weisstein, *Affine transformation*, N.A. June 10, 2019.