# MTH 252
# Midterm Review

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1. Find the absolute extrema of $f(x) = \dfrac{x}{x^2 - x + 1}$ on the interval $[0, 3]$.

> **Solution:**
>
> $$\begin{aligned} f'(x) &= \frac{(x^2 - x + 1)(1) - (x)(2x - 1)}{(x^2 - x + 1)^2} \\ &= -\frac{x^2 - 1}{(x^2 - x + 1)^2} \\ &= -\frac{(x + 1)(x - 1)}{(x^2 - x + 1)^2} \end{aligned}$$
>
> The only critical value is 1, since $-1 \notin [0, 3]$.
>
> $$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(3) &= \frac{3}{7} \end{aligned}$$
>
> Therefore, the absolute maximum is 1, and the absolute minimum is 0.

2. Given $f(x) = \dfrac{x}{x^2 + 1}$, find
   (a) The intervals of increase and decrease
   (b) The local extrema
   (c) The intervals of concavity
   (d) The point(s) of inflection

> **Solution:**
>
> $$\begin{aligned} f'(x) &= \frac{(x^2 + 1)(1) - (x)(2x)}{(x^2 + 1)^2} \\ &= \frac{x^2 - 2x^2 + 1}{(x^2 + 1)^2} \\ &= -\frac{x^2 - 1}{(x^2 + 1)^2} \\ &= -\frac{(x + 1)(x - 1)}{(x^2 + 1)^2} \end{aligned}$$

So the critical numbers are $-1$ and $1$. Moreover,

$$\begin{aligned} f''(x) &= \frac{(x^2+1)^2(-2x) - (-x^2+1)(2(x^2+1)(2x))}{(x^2+1)^4} \\ &= \frac{2x(x^2-3)}{(x^2+1)^3} \end{aligned}$$

So there are possible points of inflection at $x = 0$ and $x = \pm\sqrt{3}$.

| $x$ | | $-\sqrt{3}$ | $-1$ | $0$ | $1$ | $\sqrt{3}$ | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | $0$ | $+$ | $0$ | $-$ | |
| $f''(x)$ | $-$ | $0$ | $+$ | $0$ | $-$ | $0$ | $+$ |

Therefore,

(a) $f$ is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) $f$ has a local minimum at $-\frac{1}{2}$ and a local max at $\frac{1}{2}$.

(c) $f$ is concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$. $f$ is concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

(d) $f$ has a point of inflection at $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$, $(0, 0)$, and $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$.

3. Sketch the graph of a function $f$ satisfying all of the following properties:

(i) $f''(x) > 0$ on $(-\infty, -1), (-1, 2)$

(ii) $f''(x) < 0$ on $(2, \infty)$

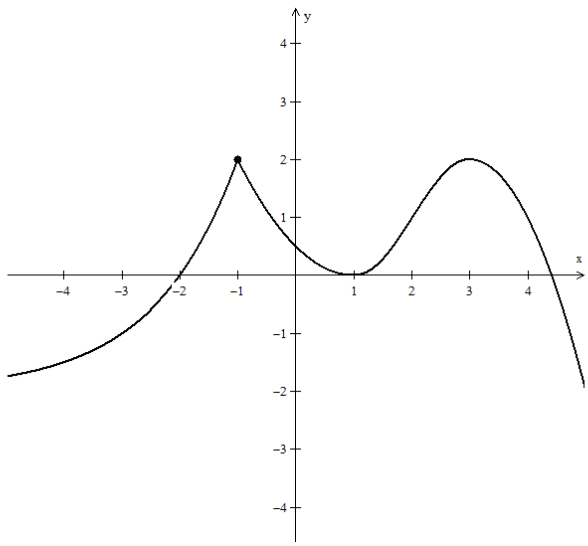
(iii) $f'(x) > 0$ on $(-\infty, -1), (1, 3)$

(iv) $f'(x) < 0$ on $(-1, 1), (3, \infty)$

(v) $f'(-1)$ does not exist

(vi) $f(-1) = 2$

**Solution:** The graph below satisfies the given conditions.




4. Find $\lim_{x\to 0} \dfrac{\sin 4x}{\tan 5x}$.

**Solution:** Since $\lim_{x\to 0} \sin 4x = \lim_{x\to 0} \tan 5x = 0$, we can use L'Hôspital.

$$\lim_{x\to 0} \frac{\sin 4x}{\tan 5x} = \lim_{x\to 0} \frac{4\cos 4x}{5\sec^2 5x} = \frac{4}{5}$$

5. Find $\lim_{x\to 0^+} \sin x \ln x$.

**Solution:** Since $\lim_{x\to 0^+} \sin x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$, we have an indeterminate form of type $0 \cdot \infty$. We can rewrite and use L'Hôspital.

$$\begin{aligned}\lim_{x\to 0^+} \frac{\ln x}{\csc x} &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} \\ &= \lim_{x\to 0^+} \frac{-\sin x \tan x}{x} \qquad \frac{0}{0} \\ &= \lim_{x\to 0^+} \frac{-\cos x \tan x - \sin x \sec^2 x}{1} \\ &= 0\end{aligned}$$

6. Find $\lim_{x\to 0} \dfrac{e^{4x} - 1 - 4x}{x^2}$.

**Solution:** This has type $\dfrac{0}{0}$, so

$$\begin{aligned}\lim_{x\to 0} \frac{e^{4x} - 1 - 4x}{x^2} &= \lim_{x\to 0} \frac{4e^{4x} - 4}{2x} \qquad \frac{0}{0} \\ &= \lim_{x\to 0} \frac{16e^{4x}}{2} \\ &= \frac{16}{2} = 8\end{aligned}$$

7. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of its base is twice the width. Material for its base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

**Solution:** The volume of a rectangular prism is $V = \ell wh = (2w)wh = 2w^2h$. Since the volume is 10 m$^2$, $10 = 2w^2h$, so $5 = w^2h$.

The surface area of the container is

$$\begin{aligned} S &= \underbrace{2wh + 2w\ell}_{\text{sides}} + \underbrace{w\ell}_{\text{base}} \\ &= 2h(w+\ell) + \ell w \\ &= 2h(w+2w) + (2w)w \\ &= 6wh + 2w^2 \end{aligned}$$

Since $5 = w^2h$, $h = \dfrac{5}{w^2}$, and so

$$S = 6\left(\frac{5}{w^2}\right)w + 2w^2 = \underbrace{\frac{30}{w}}_{\text{sides}} + \underbrace{2w^2}_{\text{base}}$$

So the cost of the container is

$$\begin{aligned} C &= \frac{30}{w}(6) + (2w^2)(10) \\ &= \frac{180}{w} + 20w^2 \\ \frac{dC}{dw} &= -\frac{180}{w^2} + 40w \\ &= -20\left(\frac{9}{w^2} - 2w\right) \end{aligned}$$

The critical number of $C$ is just $\sqrt[3]{\dfrac{9}{2}}$, found by solving $\dfrac{dC}{dw} = -20\left(\dfrac{9}{w^2} - 2w\right) = 0$. Using the first derivative test, we find that

| $x$ | | $\sqrt[3]{\frac{9}{2}}$ | |
|---|---|---|---|
| $f'(x)$ | $-$ | 0 | $+$ |

Since $C\left(\sqrt[3]{\frac{9}{2}}\right) \approx 163.54$, the minimal cost is \$163.54.

8. Find the most general antiderivative of $f(x) = 8x^9 - 3x^6 + 12x^3$.

**Solution:**

$$\begin{aligned} F(x) &= \frac{8}{10}x^{10} - \frac{3}{7}x^7 + \frac{12}{4}x^4 + C \\ &= \frac{4}{5}x^{10} - \frac{3}{7}x^7 + 3x^4 + C \end{aligned}$$

9. Find the most general antiderivative of $f(t) = \sin t + 2\cos t$.

**Solution:** $F(t) = -\cos t - 2\sin t + C$

10. Find $f$ if $f'(t) = 5t^4 - 3t^2 + 4$ and $f(-1) = 2$.

**Solution:**

$$\begin{aligned} f(t) &= \frac{5}{5}t^5 - \frac{3}{3}t^3 + 4t + C \\ &= t^5 - t^3 + 4t + C \\ 2 &= f(-1) \\ &= (-1)^5 - (-1)^3 + 4(-1) + C \\ &= -4 + C \Longrightarrow C = 6 \end{aligned}$$

$$f(t) = t^5 - t^3 + 4t + 6$$

11. Find $f$ if $f''(x) = 8x^3 + 5$ and $f(1) = 0$, $f'(1) = 8$.

**Solution:**

$$\begin{aligned} f'(x) &= \frac{8}{4}x^4 + 5x + C \\ &= 2x^4 + 5x + C \\ 8 &= f'(1) \\ &= 2(1)^4 + 5(1) + C \\ &= 7 + C \Longrightarrow C = 1 \\ f'(x) &= 2x^4 + 5x + 1 \\ f(x) &= \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + \hat{C} \\ 0 &= f(1) \\ &= \frac{2}{5}(1)^5 + \frac{5}{2}(1)^2 + (1) + \hat{C} \\ &= \frac{39}{10} + \hat{C} \Longrightarrow \hat{C} = -\frac{39}{10} \\ f(x) &= \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10} \end{aligned}$$

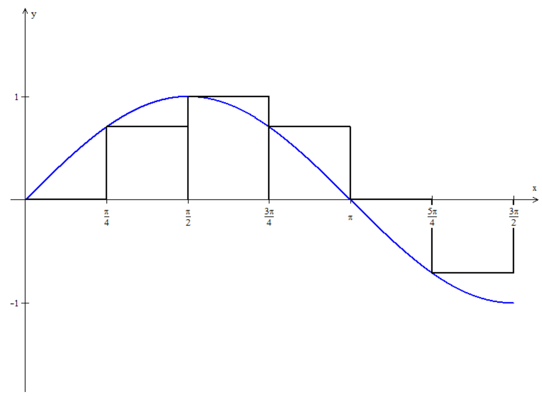
12. A particle is moving so that $a(t) = 3\cos t - 2\sin t$ with $s(0) = 0$ and $v(0) = 4$. Find the position of the particle.

**Solution:**

$$\begin{aligned} v(t) &= 3\sin t + 2\cos t + C \\ 4 &= v(0) \\ &= 3\sin 0 + 2\cos 0 + C \\ &= 2 + C \Longrightarrow C = 2 \\ v(t) &= 3\sin t + 2\cos t + 2 \\ s(t) &= -3\cos t + 2\sin t + 2t + \hat{C} \\ 0 &= s(0) \\ &= -3\cos 0 + 2\sin 0 + 2(0) + \hat{C} \\ &= -3 + \hat{C} \Longrightarrow \hat{C} = 3 \\ s(t) &= -3\cos t + 2\sin t + 2t + 3 \end{aligned}$$

13. Write a Riemann sum for $f(x) = \sin x$ on $0 \le x \le \frac{3\pi}{2}$ with six subintervals, taking sample points to be left endpoints, then find the sum.

**Solution:** Begin by graphing $y = \sin x$. Split the $x$-axis into six subintervals on $\left[0, \frac{3\pi}{2}\right]$. So $\Delta x = \dfrac{\frac{3\pi}{2} - 0}{6} = \frac{\pi}{4}$. Draw rectangles on each of those subintervals, taking heights to be vertical from the left endpoints.




Now, $n = 6$, $f(x) = \sin x$, and $x_i^* = a + i\Delta x = 0 + i\frac{\pi}{4}$, so the Riemann sum is

$$
\begin{aligned}
L_6 &= \sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{6} \sin\left(i\frac{\pi}{4}\right)\frac{\pi}{4} \\
&= \frac{\pi}{4}\left(\sin 0 + \sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4} + \sin\pi + \sin\frac{5\pi}{4}\right) \\
&= \frac{\pi}{4}\left(0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2}\right) = \frac{\pi(\sqrt{2}+2)}{8}
\end{aligned}
$$

14. Estimate $\displaystyle\int_3^9 f(x)\,dx$ with three equal subintervals using

(a) Right endpoints

(b) Left endpoints

(c) Midpoints

where values of $f(x)$ are given in the table below.

| $x$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-3.4$ | $-2.1$ | $-0.6$ | 0.3 | 0.9 | 1.4 | 1.8 |

**Solution:** For all parts, $\Delta x = \frac{b-a}{n} = \frac{9-3}{2} = 2$. With three subintervals, the endpoints are $x_0 = 3, x_1 = 5, x_2 = 7, x_3 = 9$. Then $\displaystyle\int_3^9 f(x)f(x)\,dx \approx R_3, L_3, M_3$.

$$
\begin{aligned}
R_3 &= 2(f(x_1) + f(x_2) + f(x_3)) = 2(-0.6 + 0.9 + 1.8) = 4.2 \\
L_3 &= 2(f(x_0) + f(x_1) + f(x_2)) = 2(-3.4 - 0.6 + 0.9) = -6.2 \\
M_3 &= 2\left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right)\right] = 2(f(4) + f(6) + f(8)) \\
&= 2(-2.1 + 0.3 + 1.4) = -0.8
\end{aligned}
$$

15. Evaluate $\int 3x^2 e^{-x^3}\ dx$.

**Solution:** Let $u = -x^3$, so $du = -3x^2\ dx$ and $-du = 3x^2\ dx$. Then

$$\begin{aligned}\int 3x^2 e^{-x^3}\ dx &= -\int e^u\ du \\ &= -e^u + C \\ &= -e^{-x^3} + C\end{aligned}$$

16. Evaluate $\int_0^{\frac{\pi}{4}} \sin x \sin(\cos x)\ dx$.

**Solution:** Let $u = \cos x$, so $du = -\sin x\ dx$ and $-du = \sin x\ dx$. Then our upper and lower limits of integration are $u_U = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $u_L = \cos 0 = 1$, respectively. Then

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin x \sin(\cos x)\ dx &= -\int_1^{\frac{\sqrt{2}}{2}} \sin u\ du \\ &= \cos u\Big|_1^{\frac{\sqrt{2}}{2}} \\ &= \cos\frac{\sqrt{2}}{2} - \cos 1\end{aligned}$$

17. Evaluate $\int_{-31415926}^{31415926} \frac{x^5 \sin x \tan x |x|}{12 + x^2 + x^8}\ dx$.

**Solution:** Since $12 + x^2 + x^8$ and $|x|$ are even while $x^5$, $\sin x$, and $\tan x$ are odd, our function is an odd function. Therefore,

$$\int_{-31415926}^{31415926} \frac{x^5 \sin x \tan x |x|}{12 + x^2 + x^8}\ dx = 0$$

18. Evaluate $\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\ dx$.

**Solution:** Let $u = 1 + (\ln x)^2$, $du = \frac{2\ln x}{x}\ dx$, and $\frac{1}{2}\ du = \frac{\ln x\ dx}{x}$. Then

$$\begin{aligned}\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\ dx &= \frac{1}{2}\int \frac{1}{\sqrt{u}}\ du \\ &= u^{\frac{1}{2}} + C \\ &= \sqrt{(\ln x)^2 + 1} + C\end{aligned}$$

19. Evaluate $\int \frac{3t^2 - 2}{t^3 - 2t - 8}\ dt$.

**Solution:** Let $u = t^3 - 2t - 8$. Then $du = (3t^2 - 2)\ dt$. Then

$$\begin{aligned}\int \frac{3t^2 - 2}{t^3 - 2t - 8}\ dt &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|t^3 - 2t - 8| + C\end{aligned}$$